MATH 409

Advanced Calculus I

Lecture 21:
Open and closed sets (continued).
Compact sets.

Open and closed sets

Definition. A subset $E \subset \mathbb{R}$ of the real line is called **open** if every point of E is an interior point. The subset E is called **closed** if it contains all of its limit points (or, equivalently, if it contains all of its boundary points).

Properties of open and closed sets.

- Any open interval (a, b) is an open set.
- Any closed interval $[a, b]$ is a closed set.
- If a set E is open then the complement $\mathbb{R} \setminus E$ is closed.
- If a set E is closed then the complement $\mathbb{R} \setminus E$ is open.
- The empty set and the entire real line $\mathbb{R}$ are both closed and open (in fact, these are the only sets with this property).
- Intersection of finitely many open sets is also open.
- Union of any collection of open sets is also open.
- Union of finitely many closed sets is also closed.
- Intersection of any collection of closed sets is also closed.

Interior, boundary, and closure

Recall that the set of all interior points of a set $E \subset \mathbb{R}$ is called the **interior** of E and denoted $\mathrm{int}(E)$. The set of all boundary points of E is called the **boundary** of E and denoted ∂E.

Definition. The set of all limit points of the set E, which is $\mathrm{int}(E) \cup \partial E$, is called the **closure** of E and denoted $\overline{E}$.

- The interior $\mathrm{int}(E)$ is always an open set. In other words, $\mathrm{int}(\mathrm{int}(E)) = \mathrm{int}(E)$.
- The interior $\mathrm{int}(E)$ is the largest open subset of the set E.
- The closure $\overline{E}$ is always a closed set. In other words, $\overline{\overline{E}} = \overline{E}$.
- The closure $\overline{E}$ is the smallest closed set that contains E.
- The boundary ∂E is always a closed set.
- If the set E is closed then the boundary ∂E has no interior points, that is, $\partial(\partial E) = \partial E$. In general, $\partial(\partial E) = \partial \overline{E} \subset \partial E$.

Proposition 1 Suppose x is an interior point of a set $E \subset \mathbb{R}$. Then either $(x, \infty) \subset E$ or there exists a boundary point $x_+ \in \partial E$ such that $x < x_+$ and $(x, x_+) \subset E$.

Proof: Let $S = \{y > x \mid (x, y) \subset E\}$. The set S is not empty since x is an interior point of E. If (x, ∞) is not contained in E then the set E is bounded above (since any $z > x$ not in E is an upper bound for S). Therefore $x_+ = \sup S$ is a finite number. Clearly, $x_+ > x$. Any $z \in (x, x_+)$ is not an upper bound for S; hence $y > z$ for some $y \in S$. Then $z \in (x, y) \subset E$. Thus $(x, x_+) \subset E$. At the same time, for any $y > x_+$ the interval (x, y) is not contained in E. Then $[x_+, y)$ is not contained in E as well. It follows that x_+ is a boundary point of E.

Proposition 2 Suppose x is an interior point of a set $E \subset \mathbb{R}$. Then either $(-\infty, x) \subset E$ or there exists a boundary point $x_- \in \partial E$ such that $x_- < x$ and $(x_-, x) \subset E$.

Connectedness of the real line

Corollary Between any interior point and any exterior point of a set $E \subset \mathbb{R}$, there is always a boundary point of E.

Theorem The empty set and the entire real line are the only subsets of $\mathbb{R}$ that are both open and closed.

Proof: Suppose that a set $E \subset \mathbb{R}$ is both open and closed. Then it has no boundary points. By the corollary, either all points of $\mathbb{R}$ are interior points of E or else all points of $\mathbb{R}$ are exterior points of E. In the former case, $E = \mathbb{R}$. In the latter case, E is the empty set.

General open sets

Theorem Any nonempty open set $E \subset \mathbb{R}$ decomposes as a union of a finite or infinite sequence of disjoint open intervals: $E = (a_1, b_1) \cup (a_2, b_2) \cup \dots$ Moreover, the intervals (a_i, b_i) are determined uniquely up to rearranging them.

An open interval $(a, b) \subset E$ is called a **maximal subinterval** of E if there is no other interval (c, d) such that $(a, b) \subset (c, d) \subset E$. The theorem is derived from Propositions 1 and 2 through a series of lemmas.

Lemma 1 Any point of E is contained in a maximal subinterval.

Lemma 2 Finite endpoints of a maximal subinterval do not belong to E (i.e., they are boundary points of E).

Lemma 3 Distinct maximal subintervals are disjoint.

Lemma 4 There are at most countably many maximal subintervals. (*Hint:* choose a rational point in each interval.)

Compact sets

Theorem (Bolzano-Weierstrass) Any bounded sequence of real numbers has a convergent subsequence.

Corollary Any sequence of points in a closed interval $[a, b]$ has a subsequence converging to some point in $[a, b]$.

Definition. Suppose that a set $E \subset \mathbb{R}$ has the **Bolzano-Weierstrass property**: any sequence of points from E has a subsequence converging to some point in E. Then the set E is called **compact** (or **sequentially compact**).

Theorem A set $E \subset \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof ("if"): Suppose E is closed and bounded. Then any sequence $\{x_n\}$ of points from E is bounded. Hence it has a convergent subsequence $\{x_{n_k}\}$. The limit of the subsequence is a limit point of E and so it is in E. Thus E is compact.