MATH 409

Advanced Calculus I

Lecture 22:
Limits of functions.

Limit of a function (classical definition)

Let $f : E \to \mathbb{R}$ be a function and x_0 be an interior point of its domain E.

Definition. We say that the function f **converges to a limit** $L \in \mathbb{R}$ at the point x_0 if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |x - x_0| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Notation: $L = \lim_{x \to x_0} f(x)$ or $f(x) \to L$ as $x \to x_0$.

Remark. The set $(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$ is called the **punctured δ-neighborhood** of x_0. Convergence to L means that, given $\varepsilon > 0$, the image of this set under the map f is contained in the ε-neighborhood $(L - \varepsilon, L + \varepsilon)$ of L provided that δ is small enough.

Limit of a function within a set

Let $f : E \to \mathbb{R}$ be a function and x_0 be an *accumulation* point of its domain E (x_0 may not belong to E).

Definition. We say that the function f **converges to a limit** $L \in \mathbb{R}$ at the point x_0 if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |x - x_0| < \delta \text{ and } x \in E \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

Suppose that a function $f : E_1 \to \mathbb{R}$ is defined on a set E_1 that contains E. Let $g = f|_E$ be the restriction of f to E. If the function g converges to a limit L at x_0, we say that the function f converges to L at the point x_0 **within the set** E.

Notation: $L = \lim_{\substack{x \to x_0 \\ x \in E}} f(x)$.

Limits of functions vs. limits of sequences

Theorem Let $f : E \to \mathbb{R}$ be a function and x_0 be an accumulation point of its domain E. Then $f(x) \to L$ as $x \to x_0$ if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of E different from x_0,

$$\lim_{n\to\infty} x_n = x_0 \quad \text{implies} \quad \lim_{n\to\infty} f(x_n) = L.$$

Remark. Using this sequential characterization of limits, we can derive limit theorems for convergence of functions from analogous theorems dealing with convergence of sequences.

Limits of functions vs. limits of sequences

Proof of the theorem: Suppose that $f(x) \to L$ as $x \to x_0$. Consider an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of the set $E \setminus \{x_0\}$ converging to x_0. For any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$ and $x \in E$ implies $|f(x) - L| < \varepsilon$ for all $x \in \mathbb{R}$. Further, there exists $N \in \mathbb{N}$ such that $|x_n - x_0| < \delta$ for all $n \geq N$. Then $|f(x_n) - L| < \varepsilon$ for all $n \geq N$. We conclude that $f(x_n) \to L$ as $n \to \infty$.

Conversely, suppose that $f(x) \not\to L$ as $x \to x_0$. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$ the image of a set $(x_0 - \delta, x_0 + \delta) \cap E \setminus \{x_0\}$ under the map f is not contained in $(L - \varepsilon, L + \varepsilon)$. In particular, for any $n \in \mathbb{N}$ there exists a point $x_n \in (x_0 - 1/n, x_0) \cup (x_0, x_0 + 1/n)$ such that $x_n \in E$ and $|f(x_n) - L| \geq \varepsilon$. We have that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_0 and $x_n \in E \setminus \{x_0\}$ for all $n \in \mathbb{N}$. However $f(x_n) \not\to L$ as $n \to \infty$.

Limit theorems

Squeeze Theorem Let $f, g, h : E \to \mathbb{R}$ be functions and x_0 be an accumulation point of their common domain E. If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = L$ and $f(x) \leq h(x) \leq g(x)$ for all $x \in E$, then $\lim_{x \to x_0} h(x) = L$.

Comparison Theorem If $\lim_{x \to x_0} f(x) = L$, $\lim_{x \to x_0} g(x) = M$, and $f(x) \leq g(x)$ for all x in a set for which x_0 is an accumulation point, then $L \leq M$.

Limit theorems

Theorem Let $f, g : E \to \mathbb{R}$ be functions and x_0 be an accumulation point of their common domain E. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then

$$\lim_{x \to x_0} (f + g)(x) = L + M,$$

$$\lim_{x \to x_0} (f - g)(x) = L - M,$$

$$\lim_{x \to x_0} (fg)(x) = LM.$$

If, additionally, $M \neq 0$ then

$$\lim_{x \to x_0} (f/g)(x) = L/M.$$

Divergence to infinity

Let $f : E \to \mathbb{R}$ be a function and x_0 be an accumulation point of its domain E.

Definition. We say that the function f **diverges to** $+\infty$ at the point x_0 if for every $C \in \mathbb{R}$ there exists $\delta = \delta(C) > 0$ such that

$$0 < |x - x_0| < \delta \text{ and } x \in E \text{ implies } f(x) > C.$$

Notation: $\lim_{x \to x_0} f(x) = +\infty$ or $f(x) \to +\infty$ as $x \to x_0$.

Similarly, we define the **divergence to** $-\infty$ at the point x_0.

One-sided limits

Let $f : E \to \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$.

Definition. We say that f **converges to a right-hand limit** $L \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if x_0 is an accumulation point of the set $E \cap (x_0, \infty)$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x_0 < x < x_0 + \delta$ and $x \in E$ implies $|f(x) - L| < \varepsilon$.

Notation: $L = \lim_{x \to x_0+} f(x)$.

Similarly, we define the **left-hand limit** $\lim_{x \to x_0-} f(x)$.

Note that $\lim_{x \to x_0+} f(x) = \lim_{\substack{x \to x_0 \\ x \in E \cap (x_0, \infty)}} f(x)$, $\lim_{x \to x_0-} f(x) = \lim_{\substack{x \to x_0 \\ x \in E \cap (-\infty, x_0)}} f(x)$.

Theorem Suppose x_0 is an accumulation point for both $E \cap (x_0, \infty)$ and $E \cap (-\infty, x_0)$. Then $f(x) \to L$ as $x \to x_0$ if and only if $\lim_{x \to x_0+} f(x) = \lim_{x \to x_0-} f(x) = L$.

Limits at infinity

Let $f : E \to \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$.

Definition. We say that f **converges to a limit** $L \in \mathbb{R}$ as $x \to +\infty$ if the domain E is unbounded above and for every $\varepsilon > 0$ there exists a real number $C = C(\varepsilon) \in \mathbb{R}$ such that

$$x > C \text{ and } x \in E \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

Notation: $L = \lim_{x \to +\infty} f(x)$ or $f(x) \to L$ as $x \to +\infty$.

Similarly, we define the **limit** $\lim_{x \to -\infty} f(x)$.

Examples

- Constant function: $f(x) = c$ for all $x \in \mathbb{R}$ and some $c \in \mathbb{R}$.

$\lim_{x \to x_0} f(x) = c$ for all $x_0 \in \mathbb{R}$. Also, $\lim_{x \to \pm\infty} f(x) = c$.

- Identity function: $f(x) = x$, $x \in \mathbb{R}$.

$\lim_{x \to x_0} f(x) = x_0$ for all $x_0 \in \mathbb{R}$. Also, $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

- Step function: $f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$

$\lim_{x \to 0+} f(x) = 1$, $\quad \lim_{x \to 0-} f(x) = 0$.

Examples

- $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \;\; f(x) = \dfrac{1}{x}.$

$\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} 1 \Big/ \lim\limits_{x \to x_0} x = 1/x_0$ for all $x_0 \neq 0$,

$\lim\limits_{x \to 0+} f(x) = +\infty$, $\lim\limits_{x \to 0-} f(x) = -\infty$. Also, $\lim\limits_{x \to \pm\infty} f(x) = 0$.

- $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \;\; f(x) = \sin \dfrac{1}{x}.$

$\lim\limits_{x \to 0+} f(x)$ does not exist since $f((0, \delta)) = [-1, 1]$ for any $\delta > 0$.

- $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \;\; f(x) = x \sin \dfrac{1}{x}.$

$\lim\limits_{x \to 0} f(x) = 0$ since $-|x| \leq f(x) \leq |x|$.