MATH 409

Advanced Calculus I

Lecture 23:
Limits of functions (continued).

Limit of a function

Let $f : E \to \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$.

[Classical definition] We say that f converges to a limit $L \in \mathbb{R}$ at an interior point x_0 of the domain E if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x\colon \ 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$

Notation: $L = \lim\limits_{x \to x_0} f(x)$ or $f(x) \to L$ as $x \to x_0$.

[General definition] The function f is eligible to have $\lim\limits_{x \to x_0} f(x)$ if x_0 is an accumulation point of the domain E.
Assuming this, we say that $f(x) \to L \in \mathbb{R}$ as $x \to x_0$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in E\colon \ 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$$

[Limit within a set] The limit of the function f at a point x_0 within a subset $E_0 \subset E$ of the domain, denoted $\lim\limits_{\substack{x \to x_0 \\ x \in E_0}} f(x)$, is $\lim\limits_{x \to x_0} f|_{E_0}(x)$, the limit of the restriction $f|_{E_0}$.

Let $f : E \to \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$.

[One-sided limits] The right-hand limit of the function f at a point x_0, denoted $\lim_{x \to x_0+} f(x)$, is the limit at x_0 within the set $E \cap (x_0, \infty)$. The left-hand limit f at a point x_0, denoted $\lim_{x \to x_0-} f(x)$, is the limit at x_0 within the set $E \cap (-\infty, x_0)$.

[Divergence to infinity / Infinite limits] To define divergence of the function f to $+\infty$ at a point x_0 instead of convergence to a finite limit L, the requirement $|f(x) - L| < \varepsilon$ is replaced by $f(x) > 1/\varepsilon$. For divergence to $-\infty$, the new requirement is $f(x) < -1/\varepsilon$.

[Limit at infinity] Function f is eligible to have $\lim_{x \to +\infty} f(x)$, resp. $\lim_{x \to -\infty} f(x)$, if the domain E is unbounded above (resp. below). In the definition, the condition $0 < |x - x_0| < \delta$ is replaced by $x > 1/\delta$ (resp. $x < -1/\delta$).

Limits of functions vs. limits of sequences

Theorem Suppose $f : E \to \mathbb{R}$ is a function eligible to have a limit $\lim_{x \to x_0} f(x)$. Then $f(x) \to L$ as $x \to x_0$ if and only if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of E different from x_0,

$$\lim_{n \to \infty} x_n = x_0 \quad \text{implies} \quad \lim_{n \to \infty} f(x_n) = L.$$

Remark. In the theorem, x_0 can be a finite number or $+\infty$ or $-\infty$. Likewise, the limit L can be a finite number or $+\infty$ or $-\infty$.

Limit theorems

Squeeze Theorem Let $f, g, h : E \to \mathbb{R}$ be functions and x_0 be an accumulation point of their common domain E. If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = L$ and $f(x) \leq h(x) \leq g(x)$ for all $x \in E$, then $\lim_{x \to x_0} h(x) = L$.

Comparison Theorem If $\lim_{x \to x_0} f(x) = L$, $\lim_{x \to x_0} g(x) = M$, and $f(x) \leq g(x)$ for all x in a set for which x_0 is an accumulation point, then $L \leq M$.

Limit theorems

Theorem Let $f, g : E \to \mathbb{R}$ be functions and x_0 be an accumulation point of their common domain E. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then

$$\lim_{x \to x_0} (f + g)(x) = L + M,$$

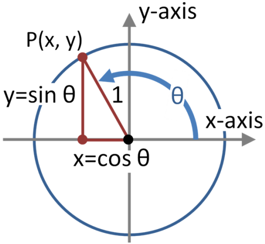
$$\lim_{x \to x_0} (f - g)(x) = L - M,$$

$$\lim_{x \to x_0} (fg)(x) = LM.$$

If, additionally, $M \neq 0$ then

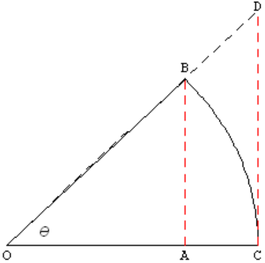
$$\lim_{x \to x_0} (f/g)(x) = L/M.$$

Trigonometric functions



$$\sin\theta = y$$
$$\cos\theta = x$$
$$\tan\theta = y/x$$

Theorem $0 \leq \sin\theta \leq \theta \leq \tan\theta$ for $\theta \in [0, \pi/2)$.



$$\sin\theta = |\text{segment } AB|$$
$$\theta = |\text{arc } CB|$$
$$\tan\theta = |\text{segment } CD|$$

Limits of trigonometric functions

- $\lim_{x \to 0} \sin x = 0.$

We know that $0 \leq \sin\theta \leq \theta$ for $\theta \in [0, \pi/2)$. Since $\sin(-\theta) = -\sin\theta$, we obtain that $-\theta \leq \sin(-\theta) \leq 0$ for $\theta \in [0, \pi/2)$. It follows that $|\sin\theta| \leq |\theta|$ whenever $|\theta| < \pi/2$. As a consequence, $\sin\theta \to 0$ as $\theta \to 0$.

- $\lim_{x \to 0} \cos^2 x = 1.$

Using a trigonometric formula $\sin^2 x + \cos^2 x = 1$, we obtain $\lim_{x \to 0} \cos^2 x = \lim_{x \to 0}(1 - \sin^2 x) = 1 - \left(\lim_{x \to 0} \sin x\right)^2 = 1 - 0^2 = 1.$

- $\lim_{x\to 0} \cos x = 1.$

We know that $0 \le \sin x \le x \le \tan x$ for $0 \le x < \pi/2$. It follows that $0 < \cos x \le 1$ for $0 < x < \pi/2$. Moreover, $\cos(-x) = \cos x$ so that $0 < \cos x \le 1$ if $0 < |x| < \pi/2$. Therefore $\cos^2 x \le \cos x \le 1$ whenever $0 < |x| < \pi/2$. Since $\lim_{x\to 0} \cos^2 x = \lim_{x\to 0} 1 = 1$, the Squeeze Theorem implies that $\cos x \to 1$ as $x \to 0$.

- $\lim_{x\to 0} \dfrac{\sin x}{x} = 1.$

We know that $0 \le \sin x \le x \le \tan x$ for $0 \le x < \pi/2$. Therefore $\cos x \le \dfrac{\sin x}{x} \le 1$ for $0 < x < \pi/2$. Since $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, the latter inequalities also hold for $-\pi/2 < x < 0$. It remains to apply the Squeeze Theorem.

Limit of the composition

Let $f : E_1 \to \mathbb{R}$ and $g : E_2 \to \mathbb{R}$ be two functions. If $f(E_1) \subset E_2$, then the composition $(g \circ f)(x) = g(f(x))$ is a well defined function on E_1.

Theorem Suppose $\lim_{x \to x_0} f(x) = y_0$ and $\lim_{y \to y_0} g(y) = L$. If, additionally, $g(y_0) = L$ or f does not take the value y_0 at all, then $\lim_{x \to x_0} (g \circ f)(x) = L$.

Proof: Consider an arbitrary sequence $\{x_n\} \subset E_1$ such that each $x_n \neq x_0$ and $x_n \to x_0$ as $n \to \infty$. We have to show that $g(f(x_n)) \to L$ as $n \to \infty$. Since $\lim_{x \to x_0} f(x) = y_0$, we obtain that $f(x_n) \to y_0$ as $n \to \infty$. Note that $f(x_n) \in E_2$. If each $f(x_n) \neq y_0$, then $\lim_{n \to \infty} g(f(x_n)) = L$ since $\lim_{y \to y_0} g(y) = L$. Otherwise $g(y_0) = L$ so that we have required convergence anyway.

Examples

- $\lim_{x\to 0} \frac{\sin 2x}{2x} = 1.$

A function $h(x) = \frac{\sin 2x}{2x}$, which is defined for $x \neq 0$, is the composition of two function, $f(x) = 2x$ and $g(x) = \frac{\sin x}{x}$, both defined for $x \neq 0$. Note that f never takes the value 0. Since $\lim_{x\to 0} f(x) = 0$ and $\lim_{y\to 0} g(y) = 1$, it follows that $\lim_{x\to 0} h(x) = \lim_{x\to 0} g(f(x)) = 1.$

- $\lim_{x\to 0} \frac{\sin 2x}{\sin x} = 2.$

$$\lim_{x\to 0} \frac{\sin 2x}{\sin x} = \lim_{x\to 0} 2\frac{\sin 2x}{2x} : \frac{\sin x}{x} = 2 \lim_{x\to 0} \frac{\sin 2x}{2x} \Big/ \lim_{x\to 0} \frac{\sin x}{x} = 2.$$

Exotic functions

- Dirichlet function: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

$\lim_{x \to x_0} f(x)$ never exists since $f((a,b)) = \{0,1\}$ for any interval (a,b). In other words, both rational and irrational points are dense in $\mathbb{R}$.

- Riemann function:

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q, \text{ a reduced fraction,} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

$\lim_{x \to x_0} f(x) = 0$ for all $x_0 \in \mathbb{R}$. Indeed, for any $n \in \mathbb{N}$ and any bounded interval (a,b), there are only finitely many points $x \in (a,b)$ such that $f(x) \geq 1/n$. On the other hand, $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ do not exist.