MATH 409

Advanced Calculus I

Lecture 24:
Continuity.
Properties of continuous functions.

Continuity

Definition. Given a set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$, and a point $c \in E$, the function f is **continuous at** c if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

We say that the function f is **continuous on** a set $E_0 \subset E$ if f is continuous at every point $c \in E_0$. The function f is **continuous** if it is continuous on the entire domain E.

Remarks. • If c is an accumulation point of E (and $c \in E$) then the function f is continuous at c if and only if $f(c) = \lim_{x \to c} f(x)$.

• If c is an isolated point of the domain E then the function f is vacuously continuous at c.

• In the case $E = [a, b]$, the function f is continuous at a if $f(a) = \lim_{x \to a+} f(x)$ and continuous at b if $f(b) = \lim_{x \to b-} f(x)$.

Examples

- Constant function: $f(x) = a$ for all $x \in \mathbb{R}$ and some $a \in \mathbb{R}$.

Since $\lim_{x \to c} f(x) = a$ for any $c \in \mathbb{R}$, the function f is continuous.

- Identity function: $f(x) = x$, $x \in \mathbb{R}$.

Since $\lim_{x \to c} f(x) = c$ for any $c \in \mathbb{R}$, the function is continuous.

- Step function: $f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$

Since $\lim_{x \to 0-} f(x) = 0$ and $\lim_{x \to 0+} f(x) = 1$, the function is not continuous at 0. It is continuous on $\mathbb{R} \setminus \{0\}$.

Theorem A function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$ if and only if for any sequence $\{x_n\}$ of elements of E, $x_n \to c$ as $n \to \infty$ implies $f(x_n) \to f(c)$ as $n \to \infty$.

Theorem Suppose that functions $f, g : E \to \mathbb{R}$ are both continuous at a point $c \in E$. Then the functions $f + g$, $f - g$, and fg are also continuous at c. If, additionally, $g(c) \neq 0$, then the function f/g is continuous at c as well.

Examples of continuous functions

- Power function: $f(x) = x^n$, $x \in \mathbb{R}$, where $n \in \mathbb{N}$.

Since the identity function is continuous and $x^{k+1} = x^k x$ for all $k \in \mathbb{N}$, it follows by induction on n that f is continuous.

- Polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$.

Since constant functions and power functions are continuous, so are the functions $f_k(x) = a_k x^k$, $x \in \mathbb{R}$. Then f is continuous as a finite sum of continuous functions.

- Rational function: $f(x) = p(x)/q(x)$, where p and q are polynomials.

Since p and q are continuous, the function f is continuous on its entire domain $\{x \in \mathbb{R} \mid q(x) \neq 0\}$.

Given a function $f : (a, b) \to \mathbb{R}$ and a point $c \in (a, b)$, let f_1 denote the restriction of f to the interval $(a, c]$ and f_2 denote the restriction of f to $[c, b)$.

Theorem The function f is continuous if and only if both restrictions f_1 and f_2 are continuous.

Proof: For any $x \in (a, c)$, the continuity of f at x is equivalent to the continuity of f_1 at x. Likewise, the continuity of f at a point $y \in (c, b)$ is equivalent to the continuity of f_2 at y. The function f is continuous at c if $f(x) \to f(c)$ as $x \to c$. The restriction f_1 is continuous at c if $f(x) \to f(c)$ as $x \to c-$. The restriction f_2 is continuous at c if $f(x) \to f(c)$ as $x \to c+$. Therefore f is continuous at c if and only if both f_1 and f_2 are continuous at c.

Example. The function $f(x) = |x|$ is continuous on $\mathbb{R}$.

Indeed, f coincides with the function $g(x) = x$ on $[0, \infty)$ and with the function $h(x) = -x$ on $(-\infty, 0]$.

Continuity of the composition

Let $f : E_1 \to \mathbb{R}$ and $g : E_2 \to \mathbb{R}$ be two functions. If $f(E_1) \subset E_2$, then the composition $(g \circ f)(x) = g(f(x))$ is a well defined function on E_1.

Theorem If f is continuous at a point $c \in E_1$ and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Proof: Let us use the sequential characterization of continuity. Consider an arbitrary sequence $\{x_n\} \subset E_1$ converging to c. We have to show that

$$(g \circ f)(x_n) \to (g \circ f)(c) \quad \text{as} \quad n \to \infty.$$

Since the function f is continuous at c, we obtain that $f(x_n) \to f(c)$ as $n \to \infty$. Moreover, all elements of the sequence $\{f(x_n)\}$ belong to the set E_2. Since the function g is continuous at $f(c)$, we obtain that $g(f(x_n)) \to g(f(c))$ as $n \to \infty$.

Examples of continuous functions

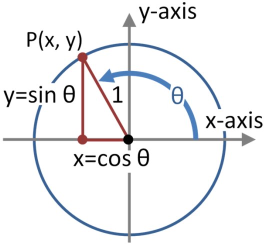
- If a function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$, then a function $g(x) = |f(x)|$, $x \in E$, is also continuous at c.

Indeed, the function g is the composition of f with the continuous function $h(x) = |x|$.

- If functions $f, g : E \to \mathbb{R}$ are continuous at a point $c \in E$, then functions $\max(f, g)$ and $\min(f, g)$ are also continuous at c.

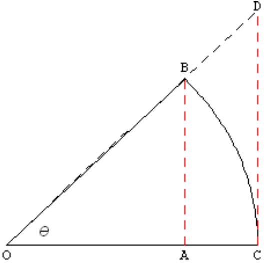
Indeed, $\max(f(x), g(x)) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$ and $\min(f(x), g(x)) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}|f(x) - g(x)|$ for all $x \in E$.

Trigonometric functions



$$\sin\theta = y$$
$$\cos\theta = x$$
$$\tan\theta = y/x$$

Theorem $0 \le \sin\theta \le \theta \le \tan\theta$ for $\theta \in [0, \pi/2)$.



$$\sin\theta = |\text{segment } AB|$$
$$\theta = |\text{arc } CB|$$
$$\tan\theta = |\text{segment } CD|$$

Examples of continuous functions

- $f(x) = \sin x, \ x \in \mathbb{R}.$

We know that $0 \leq \sin\theta \leq \theta$ for $\theta \in [0, \pi/2)$. Since $\sin(-\theta) = -\sin\theta$, we get $|\sin\theta| \leq |\theta|$ if $|\theta| < \pi/2$. In the case $|\theta| \geq \pi/2$, this estimate holds too as $|\sin\theta| \leq 1 < \pi/2$. Also, $|\cos\theta| \leq 1$ for all θ. Using the trigonometric formula

$$\sin x - \sin c = 2 \sin \tfrac{x-c}{2} \cos \tfrac{x+c}{2},$$

we obtain $|\sin x - \sin c| \leq 2\,|\sin \tfrac{x-c}{2}|\,|\cos \tfrac{x+c}{2}| \leq 2\,|\tfrac{x-c}{2}|$ $= |x - c|$. It follows that $\sin x \to \sin c$ as $x \to c$ for every $c \in \mathbb{R}$. That is, the function $\sin x$ is continuous.

- $f(x) = \cos x, \ x \in \mathbb{R}.$

Since $\cos x = \sin(x + \pi/2)$ for all $x \in \mathbb{R}$, the function f is a composition of two continuous functions, $g(x) = x + \pi/2$ and $h(x) = \sin x$. Therefore it is continuous as well.

Examples of continuous functions

- $f(x) = \tan x$.

Since $f(x) = \dfrac{\sin x}{\cos x}$, the function f is continuous on its entire domain $\mathbb{R} \setminus \{x \in \mathbb{R} \mid \cos x = 0\} = \mathbb{R} \setminus \{\pi/2 + \pi k \mid k \in \mathbb{Z}\}$.

- $f(0) = 1$ and $f(x) = \dfrac{\sin x}{x}$ for $x \neq 0$.

Since $\sin x$ and the identity function are continuous, it follows that f is continuous on $\mathbb{R} \setminus \{0\}$. Further, we know from the previous lecture that $\dfrac{\sin x}{x} \to 1$ as $x \to 0$. Thus the function f is continuous at 0 as well.

Global properties of continuous functions

Theorem 1 If $I = [a, b]$ is a closed, bounded interval of the real line, then any continuous function $f : I \to \mathbb{R}$ is bounded on I, i.e., the image $f(I)$ is a bounded subset of $\mathbb{R}$.

Theorem 2 If $I = [a, b]$ is a closed, bounded interval of the real line, then any continuous function $f : I \to \mathbb{R}$ attains its extreme values (maximum and minimum) on I.

Theorem 3 If a function $f : [a, b] \to \mathbb{R}$ is continuous then any number y_0 that lies between $f(a)$ and $f(b)$ is a value of f, i.e., $y_0 = f(x_0)$ for some $x_0 \in [a, b]$.

Extreme Value Theorem

Theorem If $I = [a, b]$ is a closed, bounded interval of the real line, then any continuous function $f : I \to \mathbb{R}$ attains its extreme values (maximum and minimum) on I. To be precise, there exist points $x_{\min}, x_{\max} \in I$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \text{ for all } x \in I.$$

Remark 1. The theorem may not hold if the interval I is not closed. Counterexample: $f(x) = x$, $x \in (0, 1)$. Neither maximum nor minimum is attained.

Remark 2. The theorem may not hold if the interval I is not bounded. Counterexample: $f(x) = 1/(1 + x^2)$, $x \in [0, \infty)$. The maximal value is attained at 0 but the minimal value is not attained.