MATH 409

Advanced Calculus I

Lecture 25:
More on continuous functions.
Points of discontinuity.

Continuity

Definition. Given a set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$, and a point $c \in E$, the function f is **continuous at** c if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

We say that the function f is **continuous on** a set $E_0 \subset E$ if f is continuous at every point $c \in E_0$. The function f is **continuous** if it is continuous on the entire domain E.

Theorem (sequential characterization of continuity) A function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$ if and only if for any sequence $\{x_n\}$ of elements of E,

$$x_n \to c \text{ as } n \to \infty \text{ implies } f(x_n) \to f(c) \text{ as } n \to \infty.$$

Continuity and compactness

Recall that a set $E \subset \mathbb{R}$ is called **compact** if it has the **Bolzano-Weierstrass property**: any sequence of points from E has a subsequence converging to some point in E.

Theorem A set $E \subset \mathbb{R}$ is compact if and only if it is closed and bounded. E.g., intervals of the form $[a, b]$ are compact.

Theorem Any continuous function maps compact sets to compact sets.

Proof: Suppose $f : E \to \mathbb{R}$ is a continuous function and $S \subset E$ is a compact subset of its domain. We need to show that the image $f(S)$ is compact as well. Let $y_1, y_2, y_3, \dots$ be an arbitrary sequence of elements of $f(S)$. For any $n \in \mathbb{N}$ we have $y_n = f(x_n)$, where $x_n \in S$. Now $x_1, x_2, x_3, \dots$ is a sequence of elements of S. Since S is compact, there is a subsequence $\{x_{n_k}\}$ converging to some $c \in S$. By continuity of f, $y_{n_k} = f(x_{n_k}) \to f(c)$, an element of $f(S)$, as $k \to \infty$.

Extreme Value Theorem

Theorem If $E \subset \mathbb{R}$ is a nonempty compact set, then any continuous function $f : E \to \mathbb{R}$ attains its extreme values (maximum and minimum) on E.

Lemma If a nonempty set $S \subset \mathbb{R}$ is bounded, then $\sup S$ and $\inf S$ are limit points of S.

Proof: We need to show that any ε-neighborhoods of the points $M = \sup S$ and $m = \inf S$ contain elements of S. Indeed, M is an upper bound for S while $M - \varepsilon$ is not. Likewise, m is a lower bound for S while $m + \varepsilon$ is not. Hence there is an element of S in $(M - \varepsilon, M]$ and in $[m, m + \varepsilon)$.

Proof of Theorem: Since E is compact and f is continuous, the image $f(E)$ is a (nonempty) compact set. Hence $f(E)$ is bounded and closed. By Lemma, $M = \sup f(E)$ and $m = \inf f(E)$ are limit points of $f(E)$. Since $f(E)$ is closed, it contains them. Thus $M = \max f(E)$ and $m = \min f(E)$.

Topological characterization of continuity

Proposition Suppose c is an interior point of a set $E \subset \mathbb{R}$. Then for any function $f : E \to \mathbb{R}$ the following are equivalent: **(i)** f is continuous at c; **(ii)** whenever $f(c)$ is an interior point of a set $U \subset \mathbb{R}$, the point c is interior for $f^{-1}(U)$.

Idea of the proof: The condition $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$ can be reformulated as $(c - \delta, c + \delta) \subset f^{-1}(U)$, where $U = (f(c) - \varepsilon, f(c) + \varepsilon)$.

Theorem Given a function $f : \mathbb{R} \to \mathbb{R}$, the following are equivalent: **(i)** f is continuous on $\mathbb{R}$; **(ii)** for any open set $U \subset \mathbb{R}$ the pre-image $f^{-1}(U)$ is also open; **(iii)** for any closed set $V \subset \mathbb{R}$ the pre-image $f^{-1}(V)$ is also closed.

Proof: Equivalence **(i)**$\Longleftrightarrow$**(ii)** follows from Proposition. Equivalence **(ii)**$\Longleftrightarrow$**(iii)** follows since the complement of an open set is closed, the complement of a closed set is open, and $f^{-1}(\mathbb{R} \setminus X) = \mathbb{R} \setminus f^{-1}(X)$ for any set $X \subset \mathbb{R}$.

Intermediate Value Theorem

Theorem If a continuous function $f : \mathbb{R} \to \mathbb{R}$ takes two different values, then it also takes all values between them.

Proof: Suppose $f(a) < f(b)$ for some $a, b \in \mathbb{R}$ and let s be any number such that $f(a) < s < f(b)$. We need to show that $f(c) = s$ for some $c \in \mathbb{R}$. Since $(-\infty, s)$ and (s, ∞) are open sets and f is a continuous function, the sets $U_- = f^{-1}((-\infty, s))$ and $U_+ = f^{-1}((s, \infty))$ are open as well. Besides, they are disjoint. Therefore any point in U_- (including a) is an interior point of U_- while any point in U_+ (including b) is an exterior point of U_-. As we know from an earlier lecture, between any interior point and any exterior point of the set U_- there must be a boundary point $c \in \partial U_-$. It is easy to see that $f(c) = s$.

Intermediate Value Theorem

Corollary 1 If a continuous function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ takes two different values, then it also takes all values between them.

Proof: Suppose $f(a) \neq f(b)$ for some $a, b \in I$, $a < b$, and let s be any number between $f(a)$ and $f(b)$. We need to show that $f(c) = s$ for some $c \in I$. Consider a function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = f(x)$ if $a \leq x \leq b$, $F(x) = f(a)$ if $x < a$, and $F(x) = f(b)$ if $x > b$. It is easy to observe that F is continuous. By the theorem, $F(c) = s$ for some $c \in \mathbb{R}$. Clearly, $a < c < b$ so that $f(c) = F(c) = s$.

Corollary 2 Any continuous function maps intervals to intervals.

Points of discontinuity

A function $f : E \to \mathbb{R}$ is **discontinuous** at a point $c \in E$ if it is not continuous at c. There are various kinds of discontinuities including the following ones.

- The function f has a **removable discontinuity** at a point c if the limit at c exists, but it is different from the value at c: $\lim_{x \to c} f(x) \neq f(c)$.

- The function f has a **jump discontinuity** at a point c if both one-sided limits at c exist, but they are not equal: $\lim_{x \to c-} f(x) \neq \lim_{x \to c+} f(x)$.

- Any other discontinuity of f is called **essential**.

- An example of an essential discontinuity is a point $c \in E$ at which the function f is **not locally bounded**, that is, f is not bounded on $(c - \delta, c + \delta) \cap E$ for any $\delta > 0$.

Half-continuity

Definition. Given a set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$, and a point $c \in E$, the function f is **continuous** c if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

The function f is **right-continuous** at c if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $c \leq x < c + \delta$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

The function f is **left-continuous** at c if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $c - \delta < x \leq c$ and $x \in E$ imply $|f(x) - f(c)| < \varepsilon$.

The function f is **upper semi-continuous** at c if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $f(x) - f(c) < \varepsilon$.

The function f is **lower semi-continuous** at c if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - c| < \delta$ and $x \in E$ imply $f(x) - f(c) > -\varepsilon$.

Examples

- Step function: $f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$

Since $\lim_{x \to 0-} f(x) = 0$ and $\lim_{x \to 0+} f(x) = 1,$ the function has a jump discontinuity at 0. It is continuous on $\mathbb{R} \setminus \{0\}$. Also, it is left-continuous and lower semi-continuous on $\mathbb{R}$.

- Integer part: $f(x) = \lfloor x \rfloor,\ x \in \mathbb{R}$.

For every integer $n \in \mathbb{Z}$, we have $f(x) = n$ if $n \le x < n+1$. It follows that $\lim_{x \to n-} f(x) = n - 1$ and $\lim_{x \to n+} f(x) = n$. Hence the function has a jump discontinuity at each integer. It is continuous on $\mathbb{R} \setminus \mathbb{Z}$. Also, it is right-continuous and upper semi-continuous on $\mathbb{R}$.

Examples

- $f(0) = 0$ and $f(x) = \dfrac{1}{x}$ for $x \neq 0$.

The function is discontinuous at 0 as it is not locally bounded at 0. It is continuous on $\mathbb{R} \setminus \{0\}$.

- $f(0) = 0$ and $f(x) = \sin \dfrac{1}{x}$ for $x \neq 0$.

Since $\lim_{x \to 0+} f(x)$ does not exist, the function is discontinuous at 0. Notice that it is an essential discontinuity, and the function f is bounded. The function f is continuous on $\mathbb{R} \setminus \{0\}$. Redefining f at 0 so that $f(0) \geq 1$, we will make the function upper semi-continuous on $\mathbb{R}$. Redefining f at 0 so that $f(0) \leq -1$, we will make it lower semi-continuous on $\mathbb{R}$.

Examples

- Dirichlet function: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Since $\lim_{x \to c} f(x)$ never exists, the function has no points of continuity. It is upper semi-continuous at rational points and lower semi-continuous at irrational points.

- Riemann function:

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q, \text{ a reduced fraction,} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Since $\lim_{x \to c} f(x) = 0$ for all $c \in \mathbb{R}$, the function f is continuous at irrational points and discontinuous at rational points. Moreover, all discontinuities are removable. Also, f is upper semi-continuous on $\mathbb{R}$.