MATH 409

Advanced Calculus I

Lecture 26:
Monotonic functions.
Exponential function.
Uniform continuity.

Monotonic functions

Let $f : E \to \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$.

Definition. The function f is called **nondecreasing** if, for any $x, y \in E$, $x < y$ implies $f(x) \leq f(y)$. It is called (strictly) **increasing** if $x < y$ implies $f(x) < f(y)$. Likewise, f is **nonincreasing** if $x < y$ implies $f(x) \geq f(y)$ and (strictly) **decreasing** $x < y$ implies $f(x) > f(y)$ for all $x, y \in E$.

Nondecreasing and nonincreasing functions are called **monotonic**. Increasing and decreasing functions are called **strictly monotonic**.

Theorem 1 Any one-sided limit of a monotonic function exists (assuming it makes sense).

Theorem 2 Any monotonic function can have only jump (or removable) discontinuities.

Theorem 3 A monotonic function f defined on an interval I is continuous if and only if the image $f(I)$ is also an interval.

Continuity of the inverse function

Suppose $f : E \to \mathbb{R}$ is a strictly monotonic function defined on a set $E \subset \mathbb{R}$. Then f is one-to-one on E so that the **inverse function** f^{-1} is a well defined function on $f(E)$.

Theorem If the domain E of a strictly monotonic function f is an interval and f is continuous on E, then the image $f(E)$ is also an interval, and the inverse function f^{-1} is strictly monotonic and continuous on $f(E)$.

Proof: Since continuous functions map intervals onto intervals, the set $f(E)$ is an interval. The inverse function f^{-1} is strictly monotonic since f is strictly monotonic. By construction, f^{-1} maps the interval $f(E)$ onto the interval E, which implies that f^{-1} is continuous.

Examples

- Power function $f(x) = x^n$, $x \in \mathbb{R}$, where $n \in \mathbb{N}$.

The function f is continuous on $\mathbb{R}$. It is strictly increasing on the interval $[0, \infty)$ and $f([0, \infty)) = [0, \infty)$. In the case n is odd, the function f is strictly increasing on $\mathbb{R}$ and $f(\mathbb{R}) = \mathbb{R}$. We conclude that the inverse function $f^{-1}(x) = \sqrt[n]{x}$ is a well defined, continuous function on $[0, \infty)$ if n is even and on $\mathbb{R}$ if n is odd.

- $f(x) = \sin x$, $x \in \mathbb{R}$.

The function f is continuous on $\mathbb{R}$. It is strictly increasing on the interval $[-\pi/2, \pi/2]$ and maps this interval onto $[-1, 1]$. Therefore the inverse function $f^{-1}(x) = \arcsin x$ is a well defined, continuous function on $[-1, 1]$.

Exponential function

First we define a^n for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$ by induction on n: $a^1 = a$ and $a^{n+1} = a^n a$ for all $n \in \mathbb{N}$.

Lemma 1 $a^{m+n} = a^m a^n$ and $a^{mn} = (a^m)^n$ for all $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$.

In the case $a \neq 0$, we also let $a^0 = 1$ and $a^{-n} = 1/a^n$ for all $n \in \mathbb{N}$.

Lemma 2 $a^{m-n} = a^m/a^n$ for all $a \neq 0$ and integers $m, n \geq 0$.

Lemma 3 $a^{m+n} = a^m a^n$ and $a^{mn} = (a^m)^n$ for all $a \neq 0$ and $m, n \in \mathbb{Z}$.

Exponential function

Lemma 4 If $m_1, m_2 \in \mathbb{Z}$ and $n_1, n_2 \in \mathbb{N}$ satisfy $m_1/n_1 = m_2/n_2$, then $\sqrt[n_1]{a^{m_1}} = \sqrt[n_2]{a^{m_2}}$ for all $a > 0$.

Now for any $a > 0$ and $r \in \mathbb{Q}$ we let $a^r = \sqrt[n]{a^m}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ are chosen so that $r = m/n$. This is well-defined due to Lemma 4.

Lemma 5 $a^{r+s} = a^r a^s$ and $a^{rs} = (a^r)^s$ for all $a > 0$ and $r, s \in \mathbb{Q}$.

Lemma 6 Suppose $a > 1$ and $r \in \mathbb{Q}$. Then $a^r > 1$ if $r > 0$, $0 < a^r < 1$ if $r < 0$, and $a^r = 1$ if $r = 0$.

Exponential function

Given $a > 0$, consider a function $f_a : \mathbb{Q} \to \mathbb{R}$ defined by $f_a(r) = a^r$, $r \in \mathbb{Q}$.

Lemma 7 The function f_a is strictly increasing if $a > 1$, strictly decreasing if $0 < a < 1$, and constant if $a = 1$.

Idea of the proof: $a^r - a^s = a^s(a^{r-s} - 1)$.

Lemma 8 $\sqrt[n]{a} \to 1$ as $n \to \infty$.

Lemma 9 $\lim\limits_{r \to c} f_a(r)$ exists for any $c \in \mathbb{R}$.

Proof: By Lemma 7, f_a is monotonic. Hence both one-sided limits $L_+ = \lim\limits_{r \to c+} f_a(r)$ and $L_- = \lim\limits_{r \to c-} f_a(r)$ exist. We need to show that $L_+ = L_-$. Let us choose a sequence $\{r_n\}$ of rational numbers such that $c - 1/n < r_n < c$ for each $n \in \mathbb{N}$. Further, let $s_n = r_n + 1/n$, $n \in \mathbb{N}$. Then $c < s_n < c + 1/n$. It follows that $a^{r_n} \to L_-$ and $a^{s_n} \to L_+$ as $n \to \infty$. Since $a^{s_n} = a^{r_n} a^{1/n}$, Lemma 8 implies that $L_+ = L_-$.

Exponential function

Given $a > 0$, for any irrational $x \in \mathbb{R} \setminus \mathbb{Q}$ we let $a^x = \lim_{\substack{r \to x \\ r \in \mathbb{Q}}} a^r$.
This is well-defined due to Lemma 9.

Now we have a function $F_a : \mathbb{R} \to \mathbb{R}$ defined by $F_a(x) = a^x$, $x \in \mathbb{R}$, namely, the **exponential function** with base a.

Theorem (i) The function F_a is strictly increasing if $a > 1$, strictly decreasing if $0 < a < 1$, and constant if $a = 1$.
(ii) The function F_a is continuous on $\mathbb{R}$.
(iii) $a^{x+y} = a^x a^y$ for all $x, y \in \mathbb{R}$.

Idea of the proof: Part (i) follows from Lemma 7. Part (ii) follows from Lemma 9 and monotonicity of F_a. Part (iii) follows from Lemma 5 and continuity of F_a.

Corollary If $a \neq 1$ then the function F_a is strictly monotonic and maps $\mathbb{R}$ onto the interval $(0, \infty)$. The inverse function $F_a^{-1}(x) = \log_a x$ is well-defined and continuous on $(0, \infty)$.

Uniform continuity

Definition. A function $f : E \to \mathbb{R}$ defined on a set $E \subset \mathbb{R}$ is called **uniformly continuous** on E if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - y| < \delta$ and $x, y \in E$ imply $|f(x) - f(y)| < \varepsilon$.

Recall that the function f is continuous at a point $y \in E$ if for every $\varepsilon > 0$ there exists $\delta = \delta(y, \varepsilon) > 0$ such that $|x - y| < \delta$ and $x \in E$ imply $|f(x) - f(y)| < \varepsilon$.

Therefore the uniform continuity of f is a stronger property than the continuity of f on E.

Examples

- Constant function $f(x) = a$ is uniformly continuous on $\mathbb{R}$.

Indeed, $|f(x) - f(y)| = 0 < \varepsilon$ for any $\varepsilon > 0$ and $x, y \in \mathbb{R}$.

- Identity function $f(x) = x$ is uniformly continuous on $\mathbb{R}$.

Since $f(x) - f(y) = x - y$, we have $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \varepsilon$.

- The sine function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

It was shown in an earlier lecture that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Therefore $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \varepsilon$.

Lipschitz functions

Definition. A function $f : E \to \mathbb{R}$ is called a **Lipschitz function** if there exists a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in E$.

- Any Lipschitz function is uniformly continuous.

Using notation of the definition, let $\delta(\varepsilon) = \varepsilon/L$, $\varepsilon > 0$.
Then $|x - y| < \delta(\varepsilon)$ implies

$$|f(x) - f(y)| \leq L|x - y| < L\delta(\varepsilon) = \varepsilon$$

for all $x, y \in E$.

• The function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ but not Lipschitz.

For any $n \in \mathbb{N}$, $|f(1/n) - f(0)| = \sqrt{1/n} = \sqrt{n}\,|1/n - 0|$. It follows that f is not Lipschitz.

Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. Suppose $|x - y| < \delta$, where $x, y \geq 0$. To estimate $|f(x) - f(y)|$, we consider two cases.

In the case $x, y \in [0, \delta)$, we use the fact that f is strictly increasing. Then $|f(x) - f(y)| < f(\delta) - f(0) = \sqrt{\delta} = \varepsilon$.

Otherwise, when $x \notin [0, \delta)$ or $y \notin [0, \delta)$, we have $\max(x, y) \geq \delta$. Then

$$|\sqrt{x} - \sqrt{y}| = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \leq \frac{|x - y|}{\sqrt{\max(x, y)}} < \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta} = \varepsilon.$$

Thus f is uniformly continuous.

- The function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Let $\varepsilon = 2$ and choose an arbitrary $\delta > 0$. Let n_δ be a natural number such that $1/n_\delta < \delta$. Further, let $x_\delta = n_\delta + 1/n_\delta$ and $y_\delta = n_\delta$. Then $|x_\delta - y_\delta| = 1/n_\delta < \delta$ while

$$f(x_\delta) - f(y_\delta) = (n_\delta + 1/n_\delta)^2 - n_\delta^2 = 2 + 1/n_\delta^2 > \varepsilon.$$

We conclude that f is not uniformly continuous.

- The function $f(x) = x^2$ is Lipschitz (and hence uniformly continuous) on any bounded interval $[a, b]$.

For any $x, y \in [a, b]$ we obtain

$$\begin{aligned} |x^2 - y^2| &= |(x+y)(x-y)| = |x+y|\,|x-y| \\ &\le (|x| + |y|)\,|x-y| \le 2\max(|a|, |b|)\,|x-y|. \end{aligned}$$

Theorem Any function continuous on a compact set E (e.g., $E = [a, b]$) is also uniformly continuous on E.

Proof: Assume that a function $f : E \to \mathbb{R}$ is not uniformly continuous on E. We have to show that f is not continuous on E. By assumption, there exists $\varepsilon > 0$ such that for any $\delta > 0$ we can find two points $x, y \in E$ satisfying $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$. In particular, for any $n \in \mathbb{N}$ there exist points $x_n, y_n \in E$ such that $|x_n - y_n| < 1/n$ while $|f(x_n) - f(y_n)| \geq \varepsilon$.

Now $\{x_n\}$ is a sequence of elements of the compact set E. Hence there is a subsequence $\{x_{n_k}\}$ converging to a limit $c \in E$. Since $x_n - 1/n < y_n < x_n + 1/n$ for all $n \in \mathbb{N}$, the subsequence $\{y_{n_k}\}$ also converges to c. However the inequalities $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$ imply that at least one of the sequences $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ is not converging to $f(c)$. It follows that the function f is not continuous at c.

Theorem Suppose that a function $f : E \to \mathbb{R}$ is uniformly continuous on E. Then it maps Cauchy sequences to Cauchy sequences, that is, for any Cauchy sequence $\{x_n\} \subset E$ the sequence $\{f(x_n)\}$ is also Cauchy.

Proof: Let $\{x_n\} \subset E$ be a Cauchy sequence. Since the function f is uniformly continuous on E, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $|x - y| < \delta$ and $x, y \in E$ imply $|f(x) - f(y)| < \varepsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists $N = N(\delta) \in \mathbb{N}$ such that $|x_n - x_m| < \delta$ for all $n, m \geq N$. Then $|f(x_n) - f(x_m)| < \varepsilon$ for all $n, m \geq N$.
We conclude that $\{f(x_n)\}$ is a Cauchy sequence.

Continuous extension

Theorem Suppose that a function $f : E \to \mathbb{R}$ is uniformly continuous. Then it can be extended to a continuous function on the closure $\overline{E}$. Moreover, the extension is unique and uniformly continuous.

Example. Let $a > 0$. Then the function $f_a : \mathbb{Q} \to \mathbb{R}$ defined by $f_a(r) = a^r$ is uniformly continuous on $[b_1, b_2] \cap \mathbb{Q}$ for any bounded interval $[b_1, b_2]$. Hence it is uniquely extended to a continuous function on $\mathbb{R}$.