MATH 409

Advanced Calculus I

Lecture 27: The derivative. Differentiability theorems.

The derivative

Definition. A function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is said to be **differentiable** at a point $x_0 \in I$ if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. The limit is denoted $f'(x_0)$ and called the **derivative** of f at x_0.

An equivalent condition is $f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Further, the one-sided limits $f'_+(x_0) = \lim_{x \to x_0+} \frac{f(x)-f(x_0)}{x-x_0}$ and $f'_-(x_0) = \lim_{x \to x_0-} \frac{f(x)-f(x_0)}{x-x_0}$ are called the **right-hand** and **left-hand derivatives** of f at x_0. One of them or both might exist even if f is not differentiable at x_0.

Remark. If $I = [a, b]$ then $f'(a)$ is essentially $f'_+(a)$ while $f'(b)$ is essentially $f'_-(b)$.

Examples

- Constant function: $f(x) = c, \;\; x \in \mathbb{R}.$

$$\frac{f(x+h) - f(x)}{h} = \frac{c - c}{h} = 0 \;\text{ for all }\; x \in \mathbb{R} \;\text{ and }\; h \neq 0.$$

Therefore $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0.$

That is, f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$.

- Identity function: $f(x) = x, \;\; x \in \mathbb{R}.$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h) - x}{h} = 1 \;\text{ for all }\; x \in \mathbb{R}, \; h \neq 0.$$

Therefore $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1.$

That is, f is differentiable on $\mathbb{R}$ and $f'(x) = 1$ for all $x \in \mathbb{R}$.

Examples

- $f(x) = \sqrt{x}, \;\; x \in [0, \infty).$

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0}$$

$$= \frac{\sqrt{x} - \sqrt{x_0}}{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})} = \frac{1}{\sqrt{x} + \sqrt{x_0}}.$$

In the case $x_0 > 0$,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}.$$

In the case $x_0 = 0$, $\displaystyle \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{1}{\sqrt{x}} = +\infty.$

Hence f is differentiable on $(0, \infty)$ and $f'(x) = 1/(2\sqrt{x})$ for all $x > 0$. It is not differentiable at 0 as $f'(0) = +\infty$.

Examples

- $f(x) = \sin x, \quad x \in \mathbb{R}.$

Since $\sin x - \sin x_0 = 2 \sin \dfrac{x - x_0}{2} \cos \dfrac{x + x_0}{2}$, we obtain

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\sin g(x, x_0)}{g(x, x_0)} \cos \frac{x + x_0}{2}, \quad \text{where } g(x, x_0) = \frac{x - x_0}{2}.$$

Note that $\lim\limits_{x \to x_0} g(x, x_0) = 0$ and $\lim\limits_{y \to 0} \dfrac{\sin y}{y} = 1$. Moreover, $g(x, x_0) \neq 0$ if $x \neq x_0$. It follows that $\lim\limits_{x \to x_0} \dfrac{\sin g(x, x_0)}{g(x, x_0)} = 1$.

Consequently, $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = \lim\limits_{x \to x_0} \cos \dfrac{x + x_0}{2} = \cos x_0$.

Thus the function f is differentiable on $\mathbb{R}$ and $f'(x) = \cos x$ for all $x \in \mathbb{R}$.

Differentiability $\implies$ continuity

Theorem If a function f is differentiable at a point c, then it is continuous at c.

Proof:

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(f(c) + \frac{f(x)-f(c)}{x-c}(x-c)\right)$$

$$= \lim_{x\to c} f(c) + \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c}(x-c)$$

$$= f(c) + f'(c)\cdot 0 = f(c).$$

Remark. Similarly, if f has a right-hand derivative at c, then $\lim_{x\to c+} f(x) = f(c)$. If f has a left-hand derivative at c, then $\lim_{x\to c-} f(x) = f(c)$.

Sum Rule and Homogeneous Rule

Theorem If functions f and g are differentiable at a point $c \in \mathbb{R}$, then the sum $f + g$ is also differentiable at c. Moreover, $(f + g)'(c) = f'(c) + g'(c)$.

Proof: $$\lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c}$$
$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = f'(c) + g'(c).$$

Theorem If a function f is differentiable at a point $c \in \mathbb{R}$, then for any $r \in \mathbb{R}$ the scalar multiple rf is also differentiable at c. Moreover, $(rf)'(c) = rf'(c)$.

Proof: $$\lim_{x \to c} \frac{(rf)(x) - (rf)(c)}{x - c} = \lim_{x \to c} r\frac{f(x) - f(c)}{x - c} = rf'(c).$$

Product Rule

Theorem If functions f and g are differentiable at a point $c \in \mathbb{R}$, then the product $f \cdot g$ is also differentiable at c. Moreover, $(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c)$.

Proof: Let I be an interval such that f and g are both defined on I and $c \in I$. For every $x \in I \setminus \{c\}$ we have

$$f(x)g(x) - f(c)g(c) = f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c) = \big(f(x) - f(c)\big)g(x) + f(c)\big(g(x) - g(c)\big).$$

Then $\frac{(f\cdot g)(x)-(f\cdot g)(c)}{x-c} = \frac{f(x)-f(c)}{x-c}g(x) + f(c)\frac{g(x)-g(c)}{x-c}$ so that

$$\lim_{x\to c} \frac{(f \cdot g)(x) - (f \cdot g)(c)}{x - c} = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x\to c} g(x) + \lim_{x\to c} f(c) \cdot \lim_{x\to c} \frac{g(x) - g(c)}{x - c} = f'(c)g(c) + f(c)g'(c).$$

Reciprocal Rule

Theorem If a function f is differentiable at a point $c \in \mathbb{R}$ and $f(c) \neq 0$, then the function $1/f$ is also differentiable at c. Moreover, $(1/f)'(c) = -f'(c)/f^2(c)$.

Proof: The function f is defined on an interval I containing c. We know that f is continuous at c. Since $\varepsilon = |f(c)| > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for any $x \in J = I \cap (c - \delta, c + \delta)$. Then $f(x) \neq 0$ for all $x \in J$. In particular, the function $1/f$ is well defined on the interval J containing c. Now

$$\lim_{x \to c} \frac{(1/f)(x) - (1/f)(c)}{x - c} = \lim_{x \to c} \left(\frac{1}{f(x)} - \frac{1}{f(c)} \right) \frac{1}{x - c}$$

$$= \lim_{x \to c} \frac{f(c) - f(x)}{f(x)f(c)} \cdot \frac{1}{x - c} = \lim_{x \to c} \left(-\frac{f(x) - f(c)}{x - c} \cdot \frac{1}{f(x)f(c)} \right)$$

$$= -\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} \frac{1}{f(x)f(c)} = -\frac{f'(c)}{f^2(c)}.$$

Difference Rule and Quotient Rule

Theorem If functions f and g are differentiable at a point $c \in \mathbb{R}$, then the difference $f - g$ is also differentiable at c. Moreover, $(f-g)'(c) = f'(c) - g'(c)$.

Proof: By the Homogeneous Rule, the function $-g = (-1)g$ is differentiable at c and $(-g)'(c) = -g'(c)$. By the Sum Rule, the function $f - g = f + (-g)$ is also differentiable at c and $(f-g)'(c) = f'(c) + (-g)'(c) = f'(c) - g'(c)$.

Theorem If functions f and g are differentiable at $c \in \mathbb{R}$ and $g(c) \neq 0$, then the quotient f/g is also differentiable at c. Moreover, $(\frac{f}{g})'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$.

Proof: By the Reciprocal Rule, the function $1/g$ is differentiable at c and $(1/g)'(c) = -g'(c)/g^2(c)$. By the Product Rule, the function $f/g = f \cdot (1/g)$ is also differentiable at c and $(f/g)'(c) = f'(c)/g(c) + f(c)(1/g)'(c)$ $= \big(f'(c)g(c) - f(c)g'(c)\big)/g^2(c)$.

Power rule: integer exponents

Theorem $(x^n)' = nx^{n-1}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof: The proof is by induction on n. In the case $n = 1$, we have $(x^1)' = x' = 1 = 1 \cdot x^0$ for all $x \in \mathbb{R}$. Now assume that $(x^n)' = nx^{n-1}$ for some $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. Using the Product Rule, we obtain $(x^{n+1})' = (x^n x)' = (x^n)'x + x^n x'$ $= nx^{n-1}x + x^n = (n+1)x^n$.

Remark. The theorem can also be proved directly using the formula $\dfrac{x^n - c^n}{x - c} = x^{n-1} + x^{n-2}c + \cdots + xc^{n-2} + c^{n-1}$.

Theorem $(x^{-n})' = -nx^{-n-1}$ for all $x \neq 0$, $n \in \mathbb{N}$.

Proof: Using the Reciprocal Rule, we obtain $(x^{-n})' = (1/x^n)' = -(x^n)'/(x^n)^2 = -nx^{n-1}/x^{2n} = -nx^{-n-1}$.