MATH 409

Advanced Calculus I

Lecture 28:
Differentiability theorems (continued).
Derivatives of elementary functions.

The derivative

Definition. A function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is said to be **differentiable** at a point $x_0 \in I$ if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. The limit (finite or infinite) is called the **derivative** of f at x_0. In the case the function f is differentiable on the entire interval I (i.e., at every point of I), we consider the derivative of f as yet another function on I.

Notation: f'. *Alternative notation:* $\dot{f}$, $\frac{df}{dx}$, $D_x f$, $f^{(1)}$.

The value of the derivative function at a point x_0 is denoted $f'(x_0)$ or $(f(x))'|_{x=x_0}$.

For example, the derivative of a function $f(x) = x^2$ at 2 can be denoted $f'(2)$ or $(x^2)'|_{x=2}$, but not $(2^2)'$.

Examples of differentiable functions

- $1' = 0$.
- $x' = 1$.
- $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$ on $(0, \infty)$.
- $(\sin x)' = \cos x$.
- $(x^2)' = 2x$.
- $\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}$ on $\mathbb{R} \setminus \{0\}$.

Differentiability theorems

Theorem If functions f and g are differentiable at a point $c \in \mathbb{R}$ (and both defined on an interval containing c), then their sum $f + g$, difference $f - g$, and product $f \cdot g$ are also differentiable at c. Moreover,

$$(f+g)'(c) = f'(c) + g'(c),$$

$$(f-g)'(c) = f'(c) - g'(c),$$

$$(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c).$$

If, additionally, $g(c) \neq 0$ then the quotient f/g is also differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Chain Rule

Theorem If a function f is differentiable at a point $c \in \mathbb{R}$ and a function g is differentiable at $f(c)$, then the composition $g \circ f$ is differentiable at c (assuming the domain of $g \circ f$ is not just $\{c\}$). Moreover, $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Proof: If the domain of $g \circ f$ is not just $\{c\}$, then it contains an interval I such that $c \in I$ (since f is continuous at c and g is continuous at $f(c)$). Let E denote the set of all points $x \in I$ such that $f(x) \neq f(c)$. If $x \in E$ then

$$\frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}.$$

Since $\lim_{x \to c} f(x) = f(c)$, the limit of the above expression at c within the set E equals $g'(f(c)) \cdot f'(c)$. In the case c is an accumulation point for $I \setminus E$, we also need to take the limit at c within $I \setminus E$. That limit is clearly 0. Fortunately, in this case we also have $f'(c) = 0$ so that $0 = g'(f(c)) \cdot f'(c)$.

Examples of differentiation

- $f(x) = \cos x, \;\; x \in \mathbb{R}.$

The function f can be represented as a composition $f = h \circ g$, where $g(x) = x + \pi/2$ and $h(x) = \sin x$, $x \in \mathbb{R}$. Since $g'(x) = 1$ and $h'(x) = \cos x$ for all $x \in \mathbb{R}$, the Chain Rule implies that f is differentiable on $\mathbb{R}$ and $f'(x) = h'(g(x))g'(x) = \cos(x + \pi/2) = -\sin x$ for all $x \in \mathbb{R}$.

- $f(x) = \tan x, \;\; x \in (-\pi/2, \pi/2).$

Since $f(x) = \sin x / \cos x$ and $\cos x \neq 0$ for $x \in (-\pi/2, \pi/2)$, the Quotient Rule implies that f is differentiable on $(-\pi/2, \pi/2)$ and

$$f'(x) = \frac{(\sin x)' \cos x - \sin x\,(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

for all $x \in (-\pi/2, \pi/2)$.

Derivative of the inverse function

Theorem Suppose f is an invertible continuous function. If f is differentiable at a point x_0 and $f'(x_0) \neq 0$, then the inverse function is differentiable at the point $y_0 = f(x_0)$ and, moreover,

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Remark. In the case $f'(x_0) = 0$, the inverse function f^{-1} is not differentiable at $f(x_0)$. Indeed, if f^{-1} is differentiable at $y_0 = f(x_0)$, then the Chain Rule implies that

$$(f^{-1} \circ f)'(x_0) = (f^{-1})'(y_0) \cdot f'(x_0).$$

Obviously, $f^{-1} \circ f$ is the identity function. Therefore $(f^{-1} \circ f)'(x_0) = 1 \neq 0$ so that $f'(x_0) \neq 0$.

Proof of the theorem: The function f is defined on an interval I containing x_0. Since f is continuous and invertible, it follows from the Intermediate Value Theorem that f is strictly monotonic on I, the image $f(I)$ is an interval containing y_0, and the inverse function f^{-1} is strictly monotonic and continuous on $f(I)$.

We have $\displaystyle\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$. Since $f'(x_0) \neq 0$, it follows that $\displaystyle\lim_{x\to x_0} \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{f'(x_0)}$. Since f^{-1} is continuous and strictly monotonic on the interval $f(I)$, we obtain that $\displaystyle\lim_{y\to y_0} f^{-1}(y) = x_0$ and $f^{-1}(y) \neq x_0$ if $y \neq y_0$.

Therefore $\displaystyle\lim_{y\to y_0} \frac{f^{-1}(y)-x_0}{y-y_0} = \lim_{y\to y_0} \frac{f^{-1}(y)-x_0}{f(f^{-1}(y))-y_0}$

$$= \lim_{x\to x_0} \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{f'(x_0)}.$$

Example

- $f(x) = \arccos x, \quad x \in [-1, 1]$.

The function $g(y) = \cos y$ is strictly decreasing on the interval $[0, \pi]$ and maps this interval onto $[-1, 1]$. By definition, the function $f(x) = \arccos x$ is the inverse of the restriction of g to $[0, \pi]$. Notice that $g'(0) = g'(\pi) = 0$ and $g'(y) \neq 0$ for $y \in (0, \pi)$. It follows that the function f is differentiable on $(-1, 1)$ and not differentiable at 1 and -1. Moreover, for any $x \in (-1, 1)$,

$$f'(x) = \frac{1}{g'(f(x))} = -\frac{1}{\sin(\arccos x)}.$$

Let $y = \arccos x$. We have $\sin^2 y + \cos^2 y = 1$. Besides, $\sin y > 0$ since $y \in (0, \pi)$. Consequently, $\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$. Thus $f'(x) = -\dfrac{1}{\sqrt{1 - x^2}}$.

Exponential and logarithmic functions

Theorem The sequence $x_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$ is increasing and bounded, hence convergent.

The limit is the number $e = 2.718281828\ldots$ (*"I'm forming a mnemonic to remember a constant in calculus"*).

Corollary $\lim\limits_{x\to 0} (1+x)^{1/x} = e$.

For any $a > 0$, $a \neq 1$ the exponential function $f(x) = a^x$ is strictly monotonic and continuous on $\mathbb{R}$. It maps $\mathbb{R}$ onto $(0, \infty)$. Therefore the inverse function $g(y) = \log_a y$ is strictly monotonic and continuous on $(0, \infty)$. The natural logarithm $\log_e y$ is also denoted $\log y$.

Since $(1+h)^{1/h} \to e$ as $h \to 0$, it follows that $h^{-1} \log(1+h) = \log(1+h)^{1/h} \to \log e = 1$ as $h \to 0$. In other words, $(\log y)'|_{y=1} = 1$. This implies that $(e^x)'|_{x=0} = 1$.

Examples of differentiation

- $f(x) = e^x, \;\; x \in \mathbb{R}.$

$$\frac{f(x+h)-f(x)}{h} = \frac{e^{x+h}-e^x}{h} = \frac{e^x e^h - e^x}{h} = \frac{e^x(e^h-1)}{h}$$

for all $x, h \in \mathbb{R}$. Therefore for any $x \in \mathbb{R}$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x f'(0) = e^x.$$

- $f(x) = a^x, \;\; x \in \mathbb{R}, \;$ where $a > 0.$

$f(x) = e^{\log a^x} = e^{x \log a}$ so that $f'(x) = e^{x\log a} \log a = a^x \log a.$

- $f(x) = \log x, \;\; x \in (0, \infty).$

Since f is the inverse of the function $g(y) = e^y$, we obtain
$f'(x) = 1/g'(\log x) = 1/e^{\log x} = 1/x$ for all $x > 0$.

Power rule: general case

Theorem $(x^\alpha)' = \alpha x^{\alpha-1}$ for all $x > 0$ and $\alpha \in \mathbb{R}$.

Proof: Let us fix a number $\alpha \in \mathbb{R}$ and consider a function $f(x) = x^\alpha$, $x \in (0, \infty)$. For any $x > 0$ we obtain $f(x) = e^{\log(x^\alpha)} = e^{\alpha \log x} = a^{\log x}$, where $a = e^\alpha$. Hence $f = h \circ g$, where $g(x) = \log x$, $x > 0$ and $h(y) = a^y$, $y \in \mathbb{R}$. By the Chain Rule,

$$\begin{aligned} f'(x) &= h'(g(x)) \cdot g'(x) = a^{\log x} \log a \cdot (\log x)' \\ &= f(x) \log a \cdot (\log x)' = f(x) \cdot \alpha (\log x)' \\ &= f(x) \cdot \alpha / x = x^\alpha \cdot \alpha / x = \alpha x^{\alpha-1}. \end{aligned}$$