MATH 409

Advanced Calculus I

Lecture 29: Mean value theorem.

Points of local extremum

Definition. We say that a function $f : E \to \mathbb{R}$ attains a **local maximum** at a point $c \in E$ if there exists $\varepsilon > 0$ such that $f(x) \leq f(c)$ for all $x \in E \cap (c - \varepsilon, c + \varepsilon)$. Similarly, f attains a **local minimum** at $c \in E$ if there exists $\varepsilon > 0$ such that $f(x) \geq f(c)$ for all $x \in E \cap (c - \varepsilon, c + \varepsilon)$.

Theorem (Fermat) Suppose c is a point of local extremum (maximum or minimum) of a function f. If c is an interior point of the domain and f is differentiable at c, then $f'(c)=0$.

Proof: Assume c is a point of local minimum. Since f is defined on an open interval containing c, there exists $\varepsilon > 0$ such that $f(c + h) - f(c) \geq 0$ for all h, $|h| < \varepsilon$. In the case $h > 0$, this implies $\lim\limits_{h\to 0+} \frac{f(c+h)-f(c)}{h} \geq 0$. In the case $h < 0$, this implies $\lim\limits_{h\to 0-} \frac{f(c+h)-f(c)}{h} \leq 0$. Since both limits are equal to $f'(c)$, we conclude that $f'(c) = 0$.

Rolle's Theorem

Theorem Suppose that $a, b \in \mathbb{R}$ with $a < b$. If a function f is continuous on the interval $[a, b]$, differentiable on (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Proof: By the Extreme Value Theorem, the function f attains its (absolute) maximum M and minimum m on $[a, b]$. In the case $M \neq m$, at least one of the extrema is attained at a point $c \in (a, b)$. Then $f'(c) = 0$ due to Fermat's theorem. In the case $M = m$, the function f is constant on $[a, b]$. Then $f'(c) = 0$ for all $c \in (a, b)$.

Corollary If a polynomial $P(x)$ has $k > 1$ distinct real roots, then the polynomial $P'(x)$ has at least $k-1$ distinct real roots.

Proof: Let $x_1, x_2, \dots, x_k$ be distinct real roots of $P(x)$ ordered so that $x_1 < x_2 < \cdots < x_k$. By Rolle's Theorem, the derivative $P'(x)$ has a root in each of $k - 1$ intervals (x_1, x_2), $(x_2, x_3), \dots, (x_{k-1}, x_k)$.

Intermediate Value Theorem for derivatives

Theorem Suppose that a function f is differentiable on an interval $[a, b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number that lies between $f'(a)$ and $f'(b)$, then $f'(x_0) = y_0$ for some $x_0 \in (a, b)$.

Remark. Although the function f is differentiable on $[a, b]$, the derivative f' need not be continuous on $[a, b]$. Hence the Intermediate Value Theorem for continuous functions may not apply here.

Corollary If a function $f : I \to \mathbb{R}$ defined on an interval I is differentiable everywhere on I, then the derivative f' can have only essential discontinuities.

Proof of the theorem: First we consider the case when $f'(a) < 0$, $f'(b) > 0$, and $y_0 = 0$. Since f is differentiable on $[a, b]$, it is continuous on $[a, b]$. By the Extreme Value Theorem, f attains its absolute minimum on $[a, b]$ at some point x_0. Since $f'(a) < 0$, we have $f(a + h) - f(a) < 0$ for $h > 0$ sufficiently small. Hence $x_0 \neq a$. Similarly, $f'(b) > 0$ implies that $f(b + h) - f(b) < 0$ for $h < 0$ sufficiently small. Hence $x_0 \neq b$. We obtain that $x_0 \in (a, b)$. Then $f'(x_0) = 0$ due to Fermat's theorem.

Next we consider the case when $f'(a) > 0$, $f'(b) < 0$, and $y_0 = 0$. Then the function $g = -f$ is differentiable on $[a, b]$ with $g'(a) = -f'(a) < 0$ and $g'(b) = -f'(b) > 0$. By the above, $g'(x_0) = 0$ for some $x_0 \in (a, b)$. Then $f'(x_0) = 0$.

In the general case, we consider a function $h(x) = f(x) - y_0x$. It is differentiable on $[a, b]$ and $h'(x) = f'(x) - y_0$ for all $x \in [a, b]$. It follows that 0 lies between $h'(a)$ and $h'(b)$. By the above, $h'(x_0) = 0$ for some $x_0 \in (a, b)$. Then $f'(x_0) = h'(x_0) + y_0 = y_0$.

Example

- $f(0) = 0, \;\; f(x) = x^2 \sin \dfrac{1}{x}, \;\; x \neq 0.$

Using the Product Rule and the Chain Rule, we obtain that the function f is differentiable on $\mathbb{R} \setminus \{0\}$. Moreover, for any $x \neq 0$,

$$f'(x) = \left(x^2 \sin \frac{1}{x}\right)' = (x^2)' \sin \frac{1}{x} + x^2 \left(\sin \frac{1}{x}\right)'$$
$$= 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \left(\frac{1}{x}\right)' = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

The function f is differentiable at 0 as well. Indeed,

$$\frac{f(h) - f(0)}{h} = h \sin \frac{1}{h} \to 0 \;\text{ as }\; h \to 0.$$

Notice that f is **not continuously differentiable** on $\mathbb{R}$ since f' is not continuous at 0. Namely, $\lim\limits_{x \to 0+} f'(x)$ does not exist.

Mean Value Theorem

Theorem If a function f is continuous on $[a,b]$ and differentiable on (a,b), then there is $c \in (a,b)$ such that $f(b) - f(a) = f'(c)\,(b-a)$.

Proof: Let $h_0(x) = f(a)\,\dfrac{b-x}{b-a} + f(b)\,\dfrac{x-a}{b-a}$, $x \in \mathbb{R}$.

By construction, $h_0(a) = f(a)$ and $h_0(b) = f(b)$. We observe that the function h_0 is differentiable. Moreover, $h_0'(x) = \frac{f(b)-f(a)}{b-a}$ for all $x \in \mathbb{R}$. It follows that the function $h = f - h_0$ is continuous on $[a,b]$, differentiable on (a,b), and satisfies $h(a) = h(b) = 0$. By Rolle's Theorem, $h'(c) = 0$ for some $c \in (a,b)$. We have

$$h'(c) = f'(c) - h_0'(c) = f'(c) - \frac{f(b) - f(a)}{b-a}.$$

Thus $f'(c) = (f(b) - f(a))/(b-a)$ or, equivalently, $f(b) - f(a) = f'(c)\,(b-a)$.

Monotonic functions (revisited)

Theorem Suppose that a function f is continuous on an interval $[a, b]$ and differentiable on (a, b).
(i) f is nondecreasing on $[a, b]$ if and only if $f' \geq 0$ on (a, b).
(ii) f is nonincreasing on $[a, b]$ if and only if $f' \leq 0$ on (a, b).
(iii) If $f' > 0$ on (a, b), then f is strictly increasing on $[a, b]$.
(iv) If $f' < 0$ on (a, b), then f is strictly decreasing on $[a, b]$.
(v) f is constant on $[a, b]$ if and only if $f' = 0$ on (a, b).

Proof: Let $a \leq x_1 < x_2 \leq b$. By the Mean Value Theorem, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some $c \in (x_1, x_2)$. Obviously, $f'(c) > 0$ if and only if $f(x_1) < f(x_2)$. Likewise, $f'(c) \geq 0$ if and only if $f(x_1) \leq f(x_2)$. This proves statements (iii), (iv), and the "if" part of statements (i), (ii). The "only if" part of statements (i) and (ii) follows from the Comparison Theorem for limits. Finally, statement (v) follows from statements (i) and (ii).

Examples

- $e^x > x + 1$ for all $x \neq 0$.

Consider a function $f(x) = e^x - x - 1$, $x \in \mathbb{R}$. This function is differentiable on $\mathbb{R}$ and $f'(x) = e^x - 1$ for all $x \in \mathbb{R}$. We observe that the derivative f' is strictly increasing. Since $f'(0) = 0$, we have $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. It follows that the function f is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$. As a consequence, $f(x) > f(0) = 0$ for all $x \neq 0$. Thus $e^x > x + 1$ for $x \neq 0$.

- $\log x < x - 1$ for all $x > 0$, $x \neq 1$.

By the above, $e^{x-1} > (x-1) + 1 = x$ for all $x \neq 1$. Since the natural logarithm is strictly increasing on $(0, \infty)$, it follows that $\log e^{x-1} > \log x$ for $x > 0$, $x \neq 1$. Equivalently, $\log x < x - 1$ for $x > 0$, $x \neq 1$.

Problem. Find $\min\limits_{x>0} x^x$.

The function $f(x) = x^x$ is well defined and positive on $(0, \infty)$. Hence

$$f(x) = e^{\log f(x)} = e^{\log x^x} = e^{x \log x}$$

for all $x > 0$. That is, $f(x) = g(h(x))$, where $h(x) = x \log x$ and $g(y) = e^y$. Using the Chain Rule and the Product Rule, we obtain

$$\begin{aligned} f'(x) &= e^{x \log x}(x \log x)' = x^x \Big((x)' \log x + x(\log x)' \Big) \\ &= x^x(\log x + 1). \end{aligned}$$

Since the natural logarithm is strictly increasing and $\log(1/e) = -1$, it follows that $f'(x) < 0$ for $0 < x < 1/e$ and $f'(x) > 0$ for $x > 1/e$. Hence the function f is strictly decreasing on $(0, 1/e]$ and strictly increasing on $[1/e, \infty)$. Therefore $\min\limits_{x>0} f(x) = f(1/e) = (1/e)^{1/e} = e^{-1/e}$.