MATH 409

Advanced Calculus I

Lecture 30:
L'Hôpital's rule.
Taylor's formula.

Fermat's Theorem If a function f is differentiable at an interior point c of its domain that is a point of local extremum (maximum or minimum), then $f'(c) = 0$.

Rolle's Theorem If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Mean Value Theorem If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)\,(b - a)$.

Theorem Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b). Then the following hold.

(i) f is nondecreasing on $[a, b]$ if and only if $f' \geq 0$ on (a, b).
(ii) f is nonincreasing on $[a, b]$ if and only if $f' \leq 0$ on (a, b).
(iii) If $f' > 0$ on (a, b), then f is strictly increasing on $[a, b]$.
(iv) If $f' < 0$ on (a, b), then f is strictly decreasing on $[a, b]$.
(v) f is constant on $[a, b]$ if and only if $f' = 0$ on (a, b).

Cauchy's Mean Value Theorem

Theorem If functions f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$g'(c)\left(f(b) - f(a)\right) = f'(c)\left(g(b) - g(a)\right).$$

Remarks. • The usual (Lagrange's) Mean Value Theorem is a particular case of this theorem, when $g(x) = x$.

• If $g(b) \neq g(a)$ and $g'(c) \neq 0$ then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

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$$g'(c)\big(f(b) - f(a)\big) = f'(c)\big(g(b) - g(a)\big).$$

Proof: For any $x \in [a, b]$, let

$$h(x) = f(x)\big(g(b) - g(a)\big) - g(x)\big(f(b) - f(a)\big).$$

We observe that the function h is continuous on $[a, b]$ and differentiable on (a, b). Further,

$$\begin{aligned} h(a) &= f(a)\big(g(b) - g(a)\big) - g(a)\big(f(b) - f(a)\big) \\ &= f(a)\,g(b) - g(a)\,f(b), \\ h(b) &= f(b)\big(g(b) - g(a)\big) - g(b)\big(f(b) - f(a)\big) \\ &= -f(b)\,g(a) + g(b)\,f(a). \end{aligned}$$

Hence $h(a) = h(b)$. By Rolle's Theorem, $h'(c) = 0$ for some $c \in (a, b)$. It remains to notice that

$$h'(c) = f'(c)\big(g(b) - g(a)\big) - g'(c)\big(f(b) - f(a)\big).$$

L'Hôpital's Rule

L'Hôpital's Rule helps to compute limits of quotients in those cases where limit theorems do not apply (because of an indeterminacy of the form $0/0$ or ∞/∞).

Theorem Let a be either a real number or $-\infty$ or $+\infty$. Let I be an open interval such that either $a \in I$ or a is an endpoint of I. Suppose that functions f and g are differentiable on I and that $g(x), g'(x) \neq 0$ for $x \in I \setminus \{a\}$. Suppose further that

$$\lim_{\substack{x \to a \\ x \in I}} f(x) = \lim_{\substack{x \to a \\ x \in I}} g(x) = A,$$

where $A = 0$ or $\pm\infty$. If the limit $\lim\limits_{\substack{x \to a \\ x \in I}} f'(x)/g'(x)$ exists (finite or infinite), then

$$\lim_{\substack{x \to a \\ x \in I}} \frac{f(x)}{g(x)} = \lim_{\substack{x \to a \\ x \in I}} \frac{f'(x)}{g'(x)}.$$

Remark. In fact, the theorem includes several similar rules corresponding to various kinds of limits ($\lim_{x\to a+}$, $\lim_{x\to a-}$, $\lim_{x\to a}$ for $a \in \mathbb{R}$, $\lim_{x\to +\infty}$, $\lim_{x\to -\infty}$) and the two types of indeterminacy ($0/0$ and ∞/∞).

Proof in the case $\lim\limits_{x\to a+} 0/0$*:* We extend f and g to $I \cup \{a\}$ by letting $f(a) = g(a) = 0$. By hypothesis, f and g are continuous on $I \cup \{a\}$ and differentiable on I. By the Cauchy Mean Value Theorem, for any $x \in I$ there exists $c_x \in (a, x)$ such that

$$g'(c_x)\big(f(x) - f(a)\big) = f'(c_x)\big(g(x) - g(a)\big).$$

That is, $g'(c_x)f(x) = f'(c_x)g(x)$. Since $g(c_x), g'(c_x) \neq 0$, we obtain $f(x)/g(x) = f'(c_x)/g'(c_x)$. Since $c_x \in (a, x)$, we have $c_x \to a+$ as $x \to a+$. It follows that

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to a+} \frac{f'(c_x)}{g'(c_x)} = \lim_{c\to a+} \frac{f'(c)}{g'(c)}.$$

Examples

- $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

The functions $f(x) = 1 - \cos x$ and $g(x) = x^2$ are infinitely differentiable on $\mathbb{R}$. We have $\lim_{x\to 0} f(x) = f(0) = 0$ and $\lim_{x\to 0} g(x) = g(0) = 0$.

Further, $f'(x) = \sin x$ and $g'(x) = 2x$. We obtain $\lim_{x\to 0} f'(x) = f'(0) = 0$ and $\lim_{x\to 0} g'(x) = g'(0) = 0$.

Even further, $f''(x) = \cos x$ and $g''(x) = 2$. We obtain $\lim_{x\to 0} f''(x) = f''(0) = 1$ and $\lim_{x\to 0} g''(x) = g''(0) = 2$.
It follows that $\lim_{x\to 0} f''(x)/g''(x) = 1/2$.

By l'Hôpital's Rule, $\lim_{x\to 0} f'(x)/g'(x) = 1/2$. Applying l'Hôpital's Rule once again, we obtain $\lim_{x\to 0} f(x)/g(x) = 1/2$.

Examples

- $\lim_{x\to 0+} x^\alpha \log x$ and $\lim_{x\to +\infty} x^\alpha \log x$, where $\alpha \neq 0$.

We have $\lim_{x\to 0+} \log x = -\infty$ and $\lim_{x\to +\infty} \log x = +\infty$.

Besides, $\lim_{x\to 0+} x^{-\alpha} = 0$ if $\alpha < 0$ and $+\infty$ if $\alpha > 0$.

Since $1/x \to 0+$ as $x \to +\infty$, we obtain that
$\lim_{x\to +\infty} x^{-\alpha} = \lim_{x\to 0+} x^\alpha$.

It follows that $\lim_{x\to 0+} x^\alpha \log x = -\infty$ if $\alpha < 0$ and

$\lim_{x\to +\infty} x^\alpha \log x = +\infty$ if $\alpha > 0$.

Examples

- $\lim_{x\to 0+} x^\alpha \log x$ and $\lim_{x\to +\infty} x^\alpha \log x$, where $\alpha \neq 0$.

Further, we have $x^\alpha \log x = f(x)/g(x)$, where the functions $f(x) = \log x$ and $g(x) = x^{-\alpha}$ are infinitely differentiable on $(0, \infty)$. For any $x > 0$ we obtain $f'(x) = 1/x$ and $g'(x) = -\alpha x^{-\alpha-1}$. Hence $f'(x)/g'(x) = -\alpha^{-1}x^\alpha$ for all $x > 0$. Therefore in the case $\alpha < 0$ we have $\lim_{x\to 0+} f'(x)/g'(x) = +\infty$ and $\lim_{x\to +\infty} f'(x)/g'(x) = 0$.
In the case $\alpha > 0$, the two limits are interchanged.

By l'Hôpital's Rule, $\lim_{x\to 0+} f(x)/g(x) = 0$ if $\alpha > 0$ and $\lim_{x\to +\infty} f(x)/g(x) = 0$ if $\alpha < 0$.

Taylor's formula

Theorem Let $n \in \mathbb{N}$ and $I \subset \mathbb{R}$ be an open interval. If a function $f : I \to \mathbb{R}$ is $n+1$ times differentiable on I, then for each pair of points $x, x_0 \in I$ there is a point c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Remark. The function

$$P_n^{f,x_0}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is a polynomial of degree at most n. It is called the **Taylor polynomial** of order n for the function f centered at x_0. One can check that $P_n^{f,x_0}(x_0) = f(x_0)$ and $(P_n^{f,x_0})^{(k)}(x_0) = f^{(k)}(x_0)$ for $1 \le k \le n$. Taylor's formula provides information on the remainder $R_n^{f,x_0} = f - P_n^{f,x_0}$.

Taylor's formula

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$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Remark. If the function f is infinitely differentiable at x_0, the power series

$$f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots$$

is called the **Taylor series** of f at the point x_0. The Taylor polynomials P_n^{f,x_0} are partial sums of this series. For any particular value of x, the series may or may not converge. In case of convergence, the sum may or may not be $f(x)$.

Proof of the theorem: Let us fix $x \in I$ and define functions

$$F(t) = \frac{(x-t)^{n+1}}{(n+1)!} \quad \text{and} \quad G(t) = f(x) - f(t) - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k.$$

The function F is infinitely differentiable on $\mathbb{R}$. The function G is defined and differentiable on I. By the Cauchy Mean Value Theorem, for every $x_0 \in I$, $x_0 \neq x$, there exists a point c between x_0 and x such that

$$G'(c)\bigl(F(x) - F(x_0)\bigr) = F'(c)\bigl(G(x) - G(x_0)\bigr).$$

Note that the latter follows both in the case $x_0 < x$ and in the case $x < x_0$. Clearly, $F(x) = G(x) = 0$. Further,

$$\frac{d}{dt}\left(-\frac{f^{(k)}(t)}{k!}(x-t)^k\right) = \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} - \frac{f^{(k+1)}(t)}{k!}(x-t)^k.$$

Summing up over k from 1 to n, we obtain that $G'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$. Finally, $F'(t) = -\frac{(x-t)^n}{n!}$ so that $G'(t)/F'(t) = f^{(n+1)}(t)$ for $t \neq x$. It follows that $G(x_0) = f^{(n+1)}(c)\,F(x_0)$, which implies Taylor's formula.

Examples

- $(1-x)^\alpha > 1-\alpha x$ for all $x \in (0,1)$ and $\alpha > 1$.

For any $\alpha > 0$ the function $f(x) = (1-x)^\alpha$ is infinitely differentiable on $(-\infty, 1)$. We have $f'(x) = -\alpha(1-x)^{\alpha-1}$ and $f''(x) = \alpha(\alpha-1)(1-x)^{\alpha-2}$ for all $x < 1$. Note that $f(0) = 1$ and $f'(0) = -\alpha$. By Taylor's formula, for any $x \in (0,1)$ we have

$$(1-x)^\alpha = 1 - \alpha x + \frac{\alpha(\alpha-1)(1-c)^{\alpha-2}}{2!}x^2,$$

where $0 < c < x$. If $\alpha > 1$ then $\dfrac{\alpha(\alpha-1)(1-c)^{\alpha-2}}{2!}x^2 > 0$ and the inequality follows.

Examples

- $(1-x)^\alpha < 1 - \alpha x + \dfrac{\alpha(\alpha-1)}{2}x^2$ for all $x \in (0,1)$ and $\alpha > 2$.

For any $\alpha > 0$ the function $f(x) = (1-x)^\alpha$ is infinitely differentiable on $(-\infty, 1)$. By Taylor's formula, for any $x \in (0,1)$ we have

$$(1-x)^\alpha = 1 - \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + R(x),$$

where $R(x) = \frac{f^{(3)}(c)}{3!}x^3 = -\frac{1}{3!}\alpha(\alpha-1)(\alpha-2)(1-c)^{\alpha-3}x^3$ for some $c \in (0,x)$. If $\alpha > 2$ then $R(x) < 0$ and the inequality follows.

Examples

- $$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

The function $f(x) = e^x$ is differentiable on $\mathbb{R}$ and $f' = f$ everywhere on $\mathbb{R}$. It follows by induction that f is infinitely differentiable and $f^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. In particular, $f^{(n)}(0) = 1$ for all $n \in \mathbb{N}$. Hence the above series is the Taylor series of f at 0. Consider a remainder

$$R_n(x) = e^x - 1 - x - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}.$$

By Taylor's formula, $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{e^c}{(n+1)!} x^{n+1}$, where $c = c(n, x)$ lies between 0 and x. For any fixed x, the remainder converges to 0 as $n \to \infty$.