MATH 409

Advanced Calculus I

Lecture 31:
Review for Test 2.

Topics for Test 2

Part III: Continuity

- Topology of the real line
- Limits of functions
- Continuous functions
- Uniform continuity

Thomson/Bruckner/Bruckner: 4.1–4.7, 5.1–5.2, 5.4–5.10

Topics for Test 2

Part IV-a: Differential calculus

- The derivative
- Differentiability theorems
- Mean value theorem
- L'Hôpital's rule
- Taylor's formula

Thomson/Bruckner/Bruckner: 7.1–7.7, 7.9, 7.11–7.13

Topology of the real line

Properties of points relative to a set:

- Interior point (contained in the set along with some ε-neighborhood)
- Exterior point (= interior point for the complement)
- Boundary point (= neither interior nor exterior)
- Limit point (= interior or boundary point)
- Isolated point (the only point in the set among all points in some ε-neighborhood)
- Accumulation point (= limit point and not isolated)

Properties of sets:

- Open set (all points of the set are interior)
- Closed set (contains all of its boundary points)
- Compact set (= closed and bounded)

Continuity

Theorem A function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$ if and only if for any sequence $\{x_n\}$ of elements of E, $x_n \to c$ as $n \to \infty$ implies $f(x_n) \to f(c)$ as $n \to \infty$.

Theorem Suppose that functions $f, g : E \to \mathbb{R}$ are both continuous at a point $c \in E$. Then the functions $f + g$, $f - g$, and fg are also continuous at c. If, additionally, $g(c) \neq 0$, then the function f/g is continuous at c as well.

Extreme Value Theorem If $I = [a, b]$ is a closed, bounded interval of the real line, then any continuous function $f : I \to \mathbb{R}$ is bounded and attains its extreme values (maximum and minimum) on I.

Intermediate Value Theorem If a function $f : [a, b] \to \mathbb{R}$ is continuous then any number y_0 that lies between $f(a)$ and $f(b)$ is a value of f, i.e., $y_0 = f(x_0)$ for some $x_0 \in [a, b]$.

Theorem Any function continuous on a closed bounded interval $[a, b]$ is also uniformly continuous on $[a, b]$.

Problem. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(-1) = f(0) = f(1) = 0$ and $f(x) = \dfrac{x-1}{x^2-1} \sin \dfrac{1}{x}$ for $x \in \mathbb{R} \setminus \{-1, 0, 1\}$.

(i) Determine all points at which the function f is continuous.

(ii) Is the function f uniformly continuous on the interval $(0, 1)$? Is it uniformly continuous on the interval $(1, 2)$?

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(i) Determine all points at which the function f is continuous.

The polynomial functions $g_1(x) = x - 1$ and $g_2(x) = x^2 - 1$ are continuous on the entire real line. Moreover, $g_2(x) = 0$ if and only if $x = 1$ or -1. Therefore the quotient $g(x) = g_1(x)/g_2(x)$ is well defined and continuous on $\mathbb{R} \setminus \{-1, 1\}$.

Further, the function $h_1(x) = 1/x$ is continuous on $\mathbb{R} \setminus \{0\}$. Since the function $h_2(x) = \sin x$ is continuous on $\mathbb{R}$, the composition function $h(x) = h_2(h_1(x))$ is continuous on $\mathbb{R} \setminus \{0\}$.

Clearly, $f(x) = g(x)h(x)$ for all $x \in \mathbb{R} \setminus \{-1, 0, 1\}$. It follows that the function f is continuous on $\mathbb{R} \setminus \{-1, 0, 1\}$.

It remains to determine whether the function f is continuous at points -1, 0, and 1. Observe that $g(x) = 1/(x+1)$ for all $x \in \mathbb{R} \setminus \{-1, 1\}$. Therefore $g(x) \to 1$ as $x \to 0$, $g(x) \to 1/2$ as $x \to 1$, and $g(x) \to \pm\infty$ as $x \to -1\pm$. Since the function h is continuous at -1 and 1, we have $h(x) \to h(-1) = -\sin 1$ as $x \to -1$ and $h(x) \to h(1) = \sin 1$ as $x \to 1$. Note that $\sin 1 > 0$ since $0 < 1 < \pi/2$. It follows that $f(x) \to \mp\infty$ as $x \to -1\pm$. In particular, f is discontinuous at -1.

Further, $f(x) \to \frac{1}{2}\sin 1$ as $x \to 1$. Since $f(1) = 0$, the function f has a removable discontinuity at 1.

Finally, the function f is not continuous at 0 since it has no limit at 0. To be precise, let $x_n = (\pi/2 + 2\pi n)^{-1}$ and $y_n = (-\pi/2 + 2\pi n)^{-1}$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ and $\{y_n\}$ are two sequences of positive numbers converging to 0. We have $h(x_n) = 1$ and $h(y_n) = -1$ for all $n \in \mathbb{N}$. It follows that $f(x_n) \to 1$ and $f(y_n) \to -1$ as $n \to \infty$. Hence there is no limit of $f(x)$ as $x \to 0+$.

(ii) Is the function f uniformly continuous on the interval $(0, 1)$? Is it uniformly continuous on the interval $(1, 2)$?

Any function uniformly continuous on the open interval $(0, 1)$ can be extended to a continuous function on $[0, 1]$. As a consequence, such a function has a right-hand limit at 0. However we already know that the function f has no right-hand limit at 0. Therefore f is not uniformly continuous on $(0, 1)$.

The function f is continuous on $(1, 2]$ and has a removable singularity at 1. Changing the value of f at 1 to the limit at 1, we obtain a function continuous on $[1, 2]$. It is known that every function continuous on the closed interval $[1, 2]$ is also uniformly continuous on $[1, 2]$. Further, any function uniformly continuous on the set $[1, 2]$ is also uniformly continuous on its subset $(1, 2)$. Since the redefined function coincides with f on $(1, 2)$, we conclude that f is uniformly continuous on $(1, 2)$.

Differentiability theorems

Theorem If functions f and g are differentiable at a point $c \in \mathbb{R}$, then their sum $f+g$, difference $f-g$, and product $f \cdot g$ are also differentiable at c. Moreover,

$$(f+g)'(c) = f'(c) + g'(c),$$
$$(f-g)'(c) = f'(c) - g'(c),$$
$$(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c).$$

If, additionally, $g(c) \neq 0$ then the quotient f/g is also differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Theorem If a function f is differentiable at a point $c \in \mathbb{R}$ and a function g is differentiable at $f(c)$, then the composition $g \circ f$ is differentiable at c. Moreover,

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Problem. Find the limit $\lim_{x\to 0} (1+x)^{1/x}$.

The function $f(x) = (1+x)^{1/x}$ is well defined on $(-1,0)\cup(0,\infty)$. Since $f(x) > 0$ for all $x > -1$, $x \neq 0$, a function $g(x) = \log f(x)$ is well defined on $(-1,0)\cup(0,\infty)$ as well. For any $x > -1$, $x \neq 0$, we have

$$g(x) = \log(1+x)^{1/x} = \frac{\log(1+x)}{x}.$$

Hence $g = h_1/h_2$, where the functions $h_1(x) = \log(1+x)$ and $h_2(x) = x$ are continuously differentiable on $(-1,\infty)$. Since $h_1(0) = h_2(0) = 0$, it follows that $\lim_{x\to 0} h_1(x) = \lim_{x\to 0} h_2(x) = 0$. By l'Hôpital's Rule,

$$\lim_{x\to 0} \frac{h_1(x)}{h_2(x)} = \lim_{x\to 0} \frac{h_1'(x)}{h_2'(x)}$$

assuming the latter limit exists.

Since $h_1'(0) = (1+x)^{-1}|_{x=0} = 1$ and $h_2'(0) = 1$, we obtain

$$\lim_{x \to 0} \frac{h_1(x)}{h_2(x)} = \lim_{x \to 0} \frac{h_1'(x)}{h_2'(x)} = \frac{\lim_{x \to 0} h_1'(x)}{\lim_{x \to 0} h_2'(x)} = \frac{1}{1} = 1.$$

Since $f = e^g$, the composition of g with a continuous function, it follows that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} e^{g(x)} = \exp\left(\lim_{x \to 0} g(x)\right) = e^1 = e.$$

Problem. Analyze a function $f:(-1,\infty)\to\mathbb{R}$ defined by $f(0)=e$ and $f(x)=(1+x)^{1/x}$ for $x>-1$, $x\neq 0$.

Consider a function $g(x)=\log f(x)$, $x>-1$. If $x\neq 0$, we have $g(x)=\log(1+x)/x$. Therefore g is differentiable on $(-1,0)\cup(0,\infty)$ and $g'(x)=\left(\frac{x}{1+x}-\log(1+x)\right)/x^2$ for all $x>-1$, $x\neq 0$.

Now consider another function $h:(-1,\infty)\to\mathbb{R}$ given by

$$h(x)=\frac{x}{1+x}-\log(1+x)=1-\frac{1}{1+x}-\log(1+x).$$

Note that $h(x)=x^2g'(x)$ for $x\neq 0$. The function h is differentiable on $(-1,\infty)$ and

$$h'(x)=\frac{1}{(1+x)^2}-\frac{1}{1+x}.$$

We obtain that $h'(x)>0$ for $-1<x<0$ and $h'(x)<0$ for $x>0$.

It follows that the function h is strictly increasing on $(-1, 0]$ and strictly decreasing on $[0, \infty)$. Since $h(0) = 0$, we obtain that $h(x) < 0$ for $x \neq 0$.

Then $g'(x) < 0$ for $x > -1$, $x \neq 0$ as well. Therefore the function g is strictly decreasing on $(-1, 0)$ and on $(0, \infty)$.

Since the function f is the composition of g with the strictly increasing function $y(x) = e^x$, it is also strictly decreasing on $(-1, 0)$ and on $(0, \infty)$.

Besides, we have already shown that $\lim_{x \to 0} f(x) = e$. Since $f(0) = e$, the function f is continuous at 0. It follows that f is strictly decreasing on $(-1, \infty)$.

Problem. Suppose that a function $p : \mathbb{R} \to \mathbb{R}$ is locally a polynomial, which means that for every $c \in \mathbb{R}$ there exists $\varepsilon > 0$ such that p coincides with a polynomial on the interval $(c - \varepsilon, c + \varepsilon)$. Prove that p is a polynomial.

For any $c \in \mathbb{R}$ let p_c be a polynomial and $\varepsilon_c > 0$ be a number such that $p(x) = p_c(x)$ for all $x \in (c - \varepsilon_c, c + \varepsilon_c)$. Consider the set E_0 of all points $c \in \mathbb{R}$ such that the polynomial p_c coincides with p_0.

We are going to show that E_0 is both open and closed. Since there are only two such sets, $\mathbb{R}$ and the empty set, and E_0 is clearly nonempty ($0 \in E_0$), it will follow that $E_0 = \mathbb{R}$. Consequently, $p = p_0$ on the entire real line.

Lemma If two polynomials coincide (as functions) on an open interval, then they are the same.

Proof: Suppose P and Q are two polynomials that coincide on an open interval I. Then the function $P - Q$ coincides with 0 on I. Since $P - Q$ is a polynomial and any nonzero polynomial has only finitely many roots, we conclude that $P - Q = 0$. Then the polynomials P and Q are the same.

Claim The set E_0 is both open and closed.

Proof: Suppose $c \in \mathbb{R}$ and $d \in (c - \varepsilon_c, c + \varepsilon_c)$. Then the intervals $I_c = (c - \varepsilon_c, c + \varepsilon_c)$ and $I_d = (d - \varepsilon_d, d + \varepsilon_d)$ overlap so that the intersection $I_c \cap I_d$ is also an open interval. By construction, the function p coincides with p_c on I_c and with p_d on I_d. Hence p_d coincides with p_c on $I_c \cap I_d$. By Lemma, p_d is the same as p_c. In the case $c \in E_0$, it follows that $(c - \varepsilon_c, c + \varepsilon_c) \subset E_0$. That is, every point of E_0 is an interior point. In the case $c \in \partial E_0$, we have $d \in E_0$ for some $d \in (c - \varepsilon_c, c + \varepsilon_c)$. Then $p_c = p_d = p_0$ so that $c \in E_0$.