MATH 409

Advanced Calculus I

# Lecture 32:
# Riemann integral.
# Riemann sums and Darboux sums.

# Partitions of an interval

*Definition.* A **partition** of a closed bounded interval $[a, b]$ is a finite subset $P \subset [a, b]$ that includes the endpoints $a$ and $b$.

Let $x_0, x_1, \ldots, x_n$ be the list of all elements of $P$ ordered so that $x_0 < x_1 < \cdots < x_n$ (note that $x_0 = a$ and $x_n = b$). These points split the interval $[a, b]$ into finitely many subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$. The **norm** of the partition $P$, denoted $\|P\|$, is the maximum of lengths of those subintervals: $\|P\| = \max\limits_{1 \le j \le n} |x_j - x_{j-1}|$.

Given two partitions $P$ and $Q$ of the same interval, we say that $Q$ is a **refinement** of $P$ (or that $Q$ is **finer** than $P$) if $P \subset Q$. Observe that $P \subset Q$ implies $\|Q\| \le \|P\|$.

For any two partitions $P$ and $Q$ of the interval $[a, b]$, the union $P \cup Q$ is also a partition that refines both $P$ and $Q$.

# Riemann sums and Riemann integral

*Definition.* A **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ with respect to a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\,(x_j - x_{j-1}).$$

*Remark.* $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$ if $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The norm of the partition $P$ is $\|P\| = \max_{1 \le j \le n} |x_j - x_{j-1}|$.

*Definition.* The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition $P$ and choice of samples $t_j$.

If this is the case, then the function $f$ is called **integrable** on $[a, b]$ and the limit $I(f)$ is called the **integral** of $f$ over $[a, b]$, denoted $\int_a^b f(x)\,dx$.

# Darboux sums

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of an interval $[a, b]$, where $x_0 = a < x_1 < \cdots < x_n = b$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

*Definition.* The **upper Darboux sum** (or the **upper Riemann sum**) of the function $f$ over the partition $P$ is the number

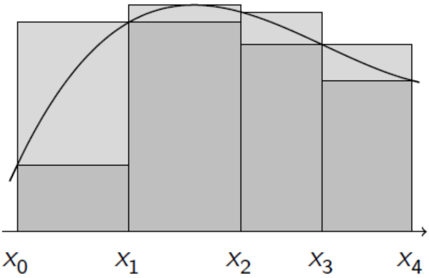
$$U(f, P) = \sum_{j=1}^{n} M_j(f)\,\Delta_j,$$

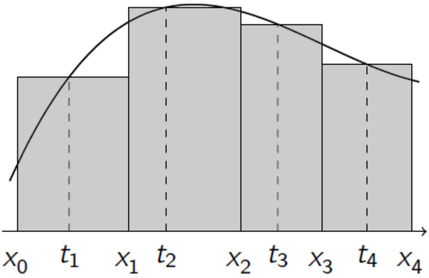
where $\Delta_j = x_j - x_{j-1}$ and $M_j(f) = \sup f([x_{j-1}, x_j])$ for $j = 1, 2, \ldots, n$. Likewise, the **lower Darboux sum** (or the **lower Riemann sum**) of $f$ over $P$ is the number

$$L(f, P) = \sum_{j=1}^{n} m_j(f)\,\Delta_j,$$

where $m_j(f) = \inf f([x_{j-1}, x_j])$ for $j = 1, 2, \ldots, n$.

# Darboux sums and a Riemann sum



# Properties of the Darboux sums

- $L(f, P) \le U(f, P)$.

Indeed, $\inf f(J) \le \sup f(J)$ for any subinterval $J \subset [a, b]$.

- $U(f, P) \le \sup f([a, b]) \cdot (b - a)$.

We have $\sup f(J) \le \sup f([a, b])$ for any subinterval $J \subset [a, b]$. Then $\sup f(J) \cdot |J| \le \sup f([a, b]) \cdot |J|$, where $|J|$ is the length of $J$. Summing up over all subintervals $J$ created by the partition $P$, we obtain $U(f, P) \le \sup f([a, b]) \cdot (b - a)$.

- $\inf f([a, b]) \cdot (b - a) \le L(f, P)$.

The proof is analogous to the previous one.

*Remark.* Observe that $\sup f([a, b]) \cdot (b - a) = U(f, P_0)$ and $\inf f([a, b]) \cdot (b - a) = L(f, P_0)$, where $P_0$ is the trivial partition: $P_0 = \{a, b\}$.

# Properties of the Darboux sums

- $L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$ for any partition $Q$ that refines $P$.

Every subinterval $J$ created by the partition $P$ is the union of one or more subintervals $J_1, J_2, \dots, J_k$ created by $Q$. Since $\sup f(J_i) \le \sup f(J)$ for $1 \le i \le k$, it follows that $\sum_{i=1}^{k} \sup f(J_i) \cdot |J_i| \le \sup f(J) \cdot \sum_{i=1}^{k} |J_i| = \sup f(J) \cdot |J|$. Summing up this inequality over all subintervals $J$, we obtain $U(f,Q) \le U(f,P)$. The inequality $L(f,P) \le L(f,Q)$ is proved in a similar way.

- $L(f,P) \le U(f,Q)$ for any partitions $P$ and $Q$.

Since the partition $P \cup Q$ refines both $P$ and $Q$, it follows from the above that $L(f,P) \le L(f,P \cup Q)$ and $U(f,P \cup Q) \le U(f,Q)$. Besides, $L(f,P \cup Q) \le U(f,P \cup Q)$.

# Upper and lower Darboux integrals

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function.

*Definition.* The **upper integral** of $f$ on $[a, b]$, denoted $\overline{\int_a^b} f(x)\,dx$ or $(U)\int_a^b f(x)\,dx$, is the number

$$\inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

Similarly, the **lower integral** of $f$ on $[a, b]$, denoted $\underline{\int_a^b} f(x)\,dx$ or $(L)\int_a^b f(x)\,dx$, is the number

$$\sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

*Remark.* Since $-\infty < L(f, P) \leq U(f, Q) < +\infty$ for all partitions $P$ and $Q$, it follows that

$$-\infty < (L)\int_a^b f(x)\,dx \leq (U)\int_a^b f(x)\,dx < +\infty.$$

# Integrability

**Proposition** If a function $f : [a, b] \to \mathbb{R}$ is integrable on the interval $[a, b]$ then it is bounded on $[a, b]$.

**Theorem 1** A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on the interval $[a, b]$ if and only if the upper and lower integrals of $f$ on $[a, b]$ coincide. If this is the case, then their common value is $\displaystyle\int_a^b f(x)\,dx$.

**Theorem 2** A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition $P_\varepsilon$ of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

*Proof of Theorem 1 ("only if"):* Assume that the Riemann sums $\mathcal{S}(f, P, t_j)$ converge to a limit $I(f)$ as $\|P\| \to 0$. Given $\varepsilon > 0$, we choose $\delta > 0$ so that for every partition $P$ with $\|P\| < \delta$, we have $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any choice of samples $t_j$. Let $\tilde{t}_j$ be a different set of samples for the same partition $P$. Then $|\mathcal{S}(f, P, \tilde{t}_j) - I(f)| < \varepsilon$. We can choose the samples $t_j, \tilde{t}_j$ so that $f(t_j)$ is arbitrarily close to $\sup f([x_{j-1}, x_j])$ while $f(\tilde{t}_j)$ is arbitrarily close to $\inf f([x_{j-1}, x_j])$. That way $\mathcal{S}(f, P, t_j)$ gets arbitrarily close to $U(f, P)$ while $\mathcal{S}(f, P, \tilde{t}_j)$ gets arbitrarily close to $L(f, P)$. Hence it follows from the above inequalities that $|U(f, P) - I(f)| \leq \varepsilon$ and $|L(f, P) - I(f)| \leq \varepsilon$. As a consequence, $|U(f, P) - L(f, P)| \leq 2\varepsilon$.

Let $I_U(f)$ and $I_L(f)$ denote the upper and lower integrals of $f$. Since $L(f, P) \leq I_L(f) \leq I_U(f) \leq U(f, P)$, it follows that $|I_U(f) - I_L(f)| \leq 2\varepsilon$. Besides, $|I_L(f) - L(f, P)| \leq 2\varepsilon$, which implies that $|I_L(f) - I(f)| \leq 3\varepsilon$. Since $\varepsilon$ can be arbitrarily small, we conclude that $I_U(f) = I_L(f) = I(f)$.

*Proof of Theorem 2:* The "if" part follows from Theorem 1 since

$$0 \leq (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx \leq U(f,P) - L(f,P)$$

for any partition $P$. Conversely, assume that $f$ is integrable on $[a,b]$. Given $\varepsilon > 0$, there exists a partition $P$ of $[a,b]$ such that

$$U(f,P) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}.$$

Also, there exists a partition $Q$ of $[a,b]$ such that

$$L(f,Q) > \int_a^b f(x)\,dx - \frac{\varepsilon}{2}.$$

Then $U(f,P) - L(f,Q) < \varepsilon$. Now $P \cup Q$ is a partition of $[a,b]$ that refines both $P$ and $Q$. It follows that $U(f,P\cup Q) \leq U(f,P)$ and $L(f,P\cup Q) \geq L(f,Q)$. Hence $U(f,P\cup Q) - L(f,P\cup Q) \leq U(f,P) - L(f,Q) < \varepsilon$.