MATH 409

Advanced Calculus I

Lecture 33:
Properties of the integral.

Riemann sums and Riemann integral

Definition. A **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ with respect to a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\,(x_j - x_{j-1}).$$

Remark. $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$ if $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The norm of the partition P is $\|P\| = \max_{1 \le j \le n} |x_j - x_{j-1}|$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

If this is the case, then the function f is called **integrable** on $[a, b]$ and the limit $I(f)$ is called the **integral** of f over $[a, b]$, denoted $\int_a^b f(x)\,dx$.

Darboux sums

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of an interval $[a, b]$, where $x_0 = a < x_1 < \cdots < x_n = b$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Definition. The **upper Darboux sum** (or the **upper Riemann sum**) of the function f over the partition P is the number

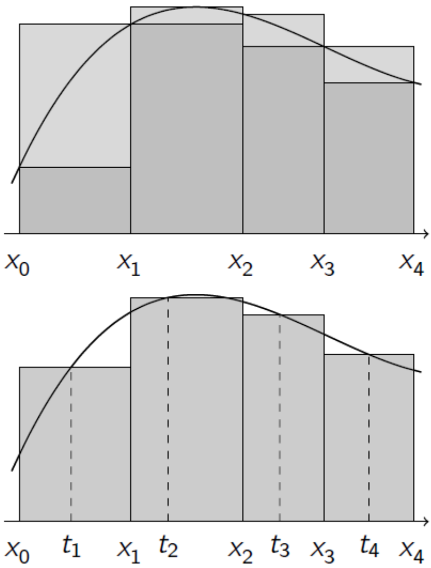
$$U(f, P) = \sum_{j=1}^{n} M_j(f)\,\Delta_j,$$

where $\Delta_j = x_j - x_{j-1}$ and $M_j(f) = \sup f([x_{j-1}, x_j])$ for $j = 1, 2, \dots, n$. Likewise, the **lower Darboux sum** (or the **lower Riemann sum**) of f over P is the number

$$L(f, P) = \sum_{j=1}^{n} m_j(f)\,\Delta_j,$$

where $m_j(f) = \inf f([x_{j-1}, x_j])$ for $j = 1, 2, \dots, n$.

Darboux sums and a Riemann sum



Upper and lower Darboux integrals

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function.

Definition. The **upper integral** of f on $[a, b]$, denoted

$\overline{\int_a^b} f(x)\, dx$ or $(U)\int_a^b f(x)\, dx$, is the number

$$\inf \{ U(f, P) \mid P \text{ is a partition of } [a, b] \}.$$

Similarly, the **lower integral** of f on $[a, b]$, denoted

$\underline{\int_a^b} f(x)\, dx$ or $(L)\int_a^b f(x)\, dx$, is the number

$$\sup \{ L(f, P) \mid P \text{ is a partition of } [a, b] \}.$$

Remark. Since $-\infty < L(f, P) \le U(f, Q) < +\infty$ for all partitions P and Q, it follows that

$$-\infty < (L)\int_a^b f(x)\, dx \le (U)\int_a^b f(x)\, dx < +\infty.$$

Integrability

Proposition If a function $f : [a, b] \to \mathbb{R}$ is integrable on the interval $[a, b]$ then it is bounded on $[a, b]$.

Theorem 1 A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on the interval $[a, b]$ if and only if the upper and lower integrals of f on $[a, b]$ coincide. If this is the case, then their common value is $\int_a^b f(x)\,dx$.

Theorem 2 A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

Examples

- Constant function $f(x) = c$ is integrable on any interval $[a, b]$ and $\int_a^b f(x)\,dx = c(b-a)$.

Indeed, for the trivial partition $P_0 = \{a, b\}$ we obtain $U(f, P_0) = c(b-a) = L(f, P_0)$.

- Step function $f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \le 0 \end{cases}$ is integrable on $[-1, 1]$ and $\int_{-1}^1 f(x)\,dx = 1$.

For any $\varepsilon \in (0, 1)$ consider a partition $P_\varepsilon = \{-1, -\varepsilon, \varepsilon, 1\}$. Then $U(f, P_\varepsilon) = 1 + \varepsilon$ and $L(f, P_\varepsilon) = 1 - \varepsilon$.

Examples

- Dirichlet function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

is not integrable on any interval $[a, b]$.

Indeed, any subinterval of $[a, b]$ contains both rational and irrational points. Therefore $U(f, P) = b - a$ and $L(f, P) = 0$ for all partitions of $[a, b]$.

- Riemann function $f(x) = \begin{cases} 1/q & \text{if } x = p/q, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

is integrable on any interval $[a, b]$.

For any $\delta > 0$ the interval $[a, b]$ contains only finitely many points $y_1, y_2, \dots, y_k$ such that $f(y_i) \geq \delta$. Let P_δ be a partition of $[a, b]$ that includes points $y_i \pm \delta/k$. Then $L(f, P_\delta) = 0$ and $U(f, P_\delta) \leq 2\delta + \delta(b - a)$.

Continuity $\implies$ integrability

Theorem If a function $f : [a, b] \to \mathbb{R}$ is continuous on the interval $[a, b]$, then it is integrable on $[a, b]$.

Proof: Since the function f is continuous, it is bounded on $[a, b]$. Furthermore, f is uniformly continuous on $[a, b]$. Therefore for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/(b - a)$ for all $x, y \in [a, b]$. Take an arbitrary partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ that satisfies $\|P\| < \delta$. Let $J = [x_{j-1}, x_j]$ be any subinterval of $[a, b]$ created by P. By the Extreme Value Theorem, there exist points $x_-, x_+ \in J$ such that $f(x_+) = \sup f(J)$ and $f(x_-) = \inf f(J)$. Since $\|P\| < \delta$, the length of J satisfies $|J| < \delta$. Then $|x_+ - x_-| \le |J| < \delta$ so that $|f(x_+) - f(x_-)| < \varepsilon/(b - a)$. It follows that $\sup f(J) \cdot |J| - \inf f(J) \cdot |J| < \varepsilon |J|/(b - a)$. Summing up the latter inequality over all subintervals J, we obtain that $U(f, P) - L(f, P) < \varepsilon$. Thus f is integrable.

Integration as a linear operation

Theorem 1 If functions f, g are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Theorem 2 If a function f is integrable on $[a, b]$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on $[a, b]$ and

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

Proof of Theorems 1 and 2: Let $I(f)$ denote the integral of f and $I(g)$ denote the integral of g over $[a, b]$. The key observation is that the Riemann sums depend linearly on a function. Namely, $\mathcal{S}(f+g, P, t_j) = \mathcal{S}(f, P, t_j) + \mathcal{S}(g, P, t_j)$ and $\mathcal{S}(\alpha f, P, t_j) = \alpha \cdot \mathcal{S}(f, P, t_j)$ for any partition P of $[a, b]$ and choice of samples t_j. It follows that

$$|\mathcal{S}(f+g, P, t_j) - I(f) - I(g)| \\ \le |\mathcal{S}(f, P, t_j) - I(f)| + |\mathcal{S}(g, P, t_j) - I(g)|,$$

$$|\mathcal{S}(\alpha f, P, t_j) - \alpha I(f)| = |\alpha| \cdot |\mathcal{S}(f, P, t_j) - I(f)|.$$

As $\|P\| \to 0$, the Riemann sums $\mathcal{S}(f, P, t_j)$ and $\mathcal{S}(g, P, t_j)$ get arbirarily close to $I(f)$ and $I(g)$, respectively. Then $\mathcal{S}(f+g, P, t_j)$ will be getting arbitrarily close to $I(f) + I(g)$ while $\mathcal{S}(\alpha f, P, t_j)$ will be getting arbitrarily close to $\alpha I(f)$. Thus $I(f) + I(g)$ is the integral of $f + g$ and $\alpha I(f)$ is the integral of αf over $[a, b]$.

Theorem If a function f is integrable on $[a, b]$, then it is integrable on each subinterval $[c, d] \subset [a, b]$.

Proof: Since f is integrable on the interval $[a, b]$, for any $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$. Given a subinterval $[c, d] \subset [a, b]$, let $P'_\varepsilon = P_\varepsilon \cup \{c, d\}$ and $Q_\varepsilon = P'_\varepsilon \cap [c, d]$. Then P'_ε is a partition of $[a, b]$ that refines P_ε. Hence

$$U(f, P'_\varepsilon) - L(f, P'_\varepsilon) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Since Q_ε is a partition of $[c, d]$ contained in P'_ε, it follows that

$$U(f, Q_\varepsilon) - L(f, Q_\varepsilon) \leq U(f, P'_\varepsilon) - L(f, P'_\varepsilon) < \varepsilon.$$

We conclude that f is integrable on $[c, d]$.

Theorem If a function f is integrable on $[a, b]$ then for any $c \in (a, b)$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof: Since f is integrable on the interval $[a, b]$, it is also integrable on subintervals $[a, c]$ and $[c, b]$. Let P be a partition of $[a, c]$ and $\{t_j\}$ be some samples for that partition. Further, let Q be a partition of $[c, b]$ and $\{\tau_i\}$ be some samples for that partition. Then $P \cup Q$ is a partition of $[a, b]$ and $\{t_j\} \cup \{\tau_i\}$ are samples for it. The key observation is that

$$\mathcal{S}(f, P \cup Q, \{t_j\} \cup \{\tau_i\}) = \mathcal{S}(f, P, t_j) + \mathcal{S}(f, Q, \tau_i).$$

If $\|P\| \to 0$ and $\|Q\| \to 0$, then $\|P \cup Q\| = \max(\|P\|, \|Q\|)$ tends to 0 as well. Therefore the Riemann sums in the latter equality will converge to the integrals $\int_a^b f(x)\,dx$, $\int_a^c f(x)\,dx$, and $\int_c^b f(x)\,dx$, respectively.

Theorem If a function f is integrable on $[a, b]$ and $f([a, b]) \subset [A, B]$, then for each continuous function $g : [A, B] \to \mathbb{R}$ the composition $g \circ f$ is also integrable on $[a, b]$.

Corollary If functions f and g are integrable on $[a, b]$, then so is fg.

Proof: We have $(f + g)^2 = f^2 + g^2 + 2fg$. Since f and g are integrable on $[a, b]$, so is $f + g$. Since $h(x) = x^2$ is a continuous function on $\mathbb{R}$, the compositions $h \circ f = f^2$, $h \circ g = g^2$, and $h \circ (f + g) = (f + g)^2$ are integrable on $[a, b]$. Then $fg = \frac{1}{2}(f + g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2$ is integrable on $[a, b]$ as a linear combination of integrable functions.

Comparison Theorem for integrals

Theorem If functions f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Proof: Since $f \leq g$ on the interval $[a, b]$, it follows that $\mathcal{S}(f, P, t_j) \leq \mathcal{S}(g, P, t_j)$ for any partition P of $[a, b]$ and choice of samples t_j. As $\|P\| \to 0$, the sum $\mathcal{S}(f, P, t_j)$ gets arbitrarily close to the integral of f while $\mathcal{S}(g, P, t_j)$ gets arbitrarily close to the integral of g. The theorem follows.

Corollary If f is integrable on $[a, b]$ and $f(x) \geq 0$ for $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$.

Sets of measure zero

Definition. A subset E of the real line $\mathbb{R}$ is said to have **measure zero** if for any $\varepsilon > 0$ the set E can be covered by countably many open intervals $J_1, J_2, \dots$ such that $\sum_{n=1}^{\infty} |J_n| < \varepsilon$.

Examples. • Any countable set has measure zero.

Indeed, suppose E is a countable set and let $x_1, x_2, \dots$ be a list of all elements of E. Given $\varepsilon > 0$, let

$$J_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right), \quad n = 1, 2, \dots$$

Then $E \subset J_1 \cup J_2 \cup \dots$ and $|J_n| = \varepsilon/2^n$ for all $n \in \mathbb{N}$ so that $\sum_{n=1}^{\infty} |J_n| = \varepsilon$.

• A nondegenerate interval $[a, b]$ is not a set of measure zero.

• There exist sets of measure zero that are of the same cardinality as $\mathbb{R}$.

Lebesgue's criterion for Riemann integrability

Definition. Suppose $P(x)$ is a property depending on $x \in S$, where $S \subset \mathbb{R}$. We say that $P(x)$ holds for **almost all** $x \in S$ (or **almost everywhere** on S) if the set $\{x \in S \mid P(x) \text{ does not hold}\}$ has measure zero.

Theorem A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on the interval $[a, b]$ if and only if f is bounded on $[a, b]$ and continuous almost everywhere on $[a, b]$.