MATH 409

Advanced Calculus I

Lecture 34: Fundamental theorem of calculus. Indefinite integral.

Integral with a variable limit

Suppose $f : [a, b] \to \mathbb{R}$ is an integrable function.

For any $x \in [a, b]$ let $F(x) = \int_a^x f(t)\, dt$

(we assume that $F(a) = 0$).

Theorem 1 The function F is well defined and continuous on $[a, b]$.

Theorem 2 If f is continuous at a point $x \in [a, b]$, then F is differentiable at x and $F'(x) = f(x)$.

Lemma If a function f is integrable on $[a, b]$, then the function $|f|$ is also integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proof: The function $|f|$ is the composition of f with a continuous function $g(x) = |x|$. Therefore $|f|$ is integrable on $[a, b]$. Since $-|f(x)| \leq f(x) \leq |f(x)|$ for $x \in [a, b]$, the Comparison Theorem for integrals implies that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

Suppose $f : [a, b] \to \mathbb{R}$ is an integrable function. For any $x \in [a, b]$ let $F(x) = \int_a^x f(t)\,dt$ (we assume that $F(a) = 0$).

Theorem 1 The function F is well defined and continuous on $[a, b]$.

Proof: Since the function f is integrable on $[a, b]$, it is also integrable on each subinterval of $[a, b]$. Hence the function F is well defined on $[a, b]$. Besides, f is bounded: $|f(t)| \leq M$ for some $M > 0$ and all $t \in [a, b]$. For any $x, y \in [a, b]$, $x \leq y$, we have $\int_a^y f(t)\,dt = \int_a^x f(t)\,dt + \int_x^y f(t)\,dt$. It follows that

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \leq \int_x^y |f(t)|\,dt \leq M\,|y - x|.$$

In other words, F is a Lipschitz function on $[a, b]$. This implies that F is uniformly continuous on $[a, b]$.

Proof of Theorem 2: For any $x, y \in [a, b]$, $x < y$, we have

$$\int_a^y f(t)\,dt = \int_a^x f(t)\,dt + \int_x^y f(t)\,dt.$$

Then

$$F(y) - F(x) - f(x)\,(y - x) = \int_x^y f(t)\,dt - \int_x^y f(x)\,dt$$

so that

$$|F(y) - F(x) - f(x)\,(y - x)| = \left| \int_x^y (f(t) - f(x))\,dt \right|$$

$$\leq \int_x^y |f(t) - f(x)|\,dt \leq \sup_{t \in [x,y]} |f(t) - f(x)|\,(y - x).$$

Finally, $\left| \dfrac{F(y) - F(x)}{y - x} - f(x) \right| \leq \sup_{t \in [x,y]} |f(t) - f(x)|.$

If the function f is right continuous at x, i.e., $f(y) \to f(x)$ as $y \to x+$, then $\sup_{t \in [x,y]} |f(t) - f(x)| \to 0$ as $y \to x+$. It follows that $f(x)$ is the right-hand derivative of F at x. Likewise, one can prove that left continuity of f at x implies that $f(x)$ is the left-hand derivative of F at x.

Fundamental theorem of calculus (part I)

Theorem If a function f is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b],$$

is continuously differentiable on $[a, b]$. Moreover, $F'(x) = f(x)$ for all $x \in [a, b]$.

Proof: Since f is continuous, it is also integrable on $[a, b]$. As already proved earlier, the integrability of f implies that the function F is well defined and continuous on $[a, b]$. Moreover, $F'(x) = f(x)$ whenever f is continuous at the point x. Therefore the continuity of f on $[a, b]$ implies that $F'(x) = f(x)$ for all $x \in [a, b]$. In particular, F is continuously differentiable on $[a, b]$.

Fundamental theorem of calculus (part II)

Theorem If a function F is differentiable on $[a, b]$ and the derivative F' is integrable on $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Remark: The derivative F' need not be continuous on $[a, b]$. Therefore Part II does not follow from Part I.

Proof: Consider any partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$. Let us choose samples $t_j \in [x_{j-1}, x_j]$ for the Riemann sum $\mathcal{S}(F', P, t_j)$ so that $F(x_j) - F(x_{j-1}) = F'(t_j)\,(x_j - x_{j-1})$ (this is possible due to the Mean Value Theorem). Then $\mathcal{S}(F', P, t_j) = \sum_{j=1}^n F'(t_j)\,(x_j - x_{j-1}) = \sum_{j=1}^n (F(x_j) - F(x_{j-1}))$ $= F(x_n) - F(x_0) = F(b) - F(a)$. Since the sums $\mathcal{S}(F', P, t_j)$ converge to $\int_a^b F'(t)\,dt$ as $\|P\| \to 0$, the theorem follows.

Indefinite integral

Definition. Given a function $f : [a, b] \to \mathbb{R}$, a function $F : [a, b] \to \mathbb{R}$ is called the **indefinite integral** (or **antiderivative**, or **primitive integral**, or **the primitive**) of f if $F'(x) = f(x)$ for all $x \in [a, b]$. Notation for F: $\int f(x)\,dx$.

If the function f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$, is an indefinite integral of f due to the Fundamental Theorem of Calculus.

Suppose F is an antiderivative of f. If G is another antiderivative of f, then $G' = F'$ on $[a, b]$. Hence $(G - F)' = G' - F' = 0$ on $[a, b]$. It follows that $G - F$ is a constant function. Conversely, for any constant C the function $G(x) = F(x) + C$ is also an antiderivative of f. Thus the general indefinite integral of f is given by

$\int f(x)\,dx = F(x) + C$, where C is an arbitrary constant.

Examples

- $\displaystyle\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + C$ on $(0,\infty)$ for $\alpha \neq -1$.

Indeed, $\displaystyle\left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = \frac{1}{\alpha+1}(x^{\alpha+1})' = \frac{1}{\alpha+1}(\alpha+1)x^\alpha = x^\alpha$.

- $\displaystyle\int \frac{1}{x} \, dx = \log|x| + C$ on $(0,\infty)$ and $(-\infty, 0)$.

Indeed, $(\log x)' = 1/x$ on $(0,\infty)$ and $(\log(-x))' = 1/x$ on $(-\infty, 0)$.

- $\displaystyle\int \sin x \, dx = -\cos x + C$.

- $\displaystyle\int \cos x \, dx = \sin x + C$.

Examples

- $\displaystyle\int \frac{x^2}{x-1}\,dx.$

To find the indefinite integral of this rational function, we expand it into the sum of a polynomial and a simple fraction:

$$\frac{x^2}{x-1} = \frac{x^2-1+1}{x-1} = \frac{x^2-1}{x-1} + \frac{1}{x-1} = x+1+\frac{1}{x-1}.$$

Since the domain of the function is $(-\infty,1)\cup(1,\infty)$, the indefinite integral has different representations on the intervals $(-\infty,1)$ and $(1,\infty)$:

$$\int \frac{x^2}{x-1}\,dx = \begin{cases} x^2/2+x+\log(x-1)+C_1, & x>1, \\ x^2/2+x+\log(1-x)+C_2, & x<1. \end{cases}$$