MATH 409

Advanced Calculus I

Lecture 35: Integration by parts. Integration by substitution.

Fundamental theorem of calculus

Theorem If a function f is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b],$$

is continuously differentiable on $[a, b]$. Moreover, $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem If a function F is differentiable on $[a, b]$ and the derivative F' is integrable on $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Linearity of the integral

Theorem If functions f, g are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Theorem If a function f is integrable on $[a, b]$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on $[a, b]$ and

$$\int_a^b \alpha f(x)\, dx = \alpha \int_a^b f(x)\, dx.$$

Integration by parts

Theorem Suppose that functions f, g are differentiable on $[a, b]$ with the derivatives f', g' integrable on $[a, b]$. Then

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

Proof: By the Product Rule, $(fg)' = f'g + fg'$ on $[a, b]$. Since the functions f, g, f', g' are integrable on $[a, b]$, so are the products $f'g$ and fg'. Then $(fg)'$ is integrable on $[a, b]$ as well. By the Fundamental Theorem of Calculus,

$$\begin{aligned} f(b)g(b) - f(a)g(a) &= \int_a^b (fg)'(x)\,dx \\ &= \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx. \end{aligned}$$

Corollary Suppose that functions f, g are continuously differentiable on $[a, b]$. Then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \quad \text{on } [a, b].$$

To simplify notation, it is convenient to use the **Leibniz differential** df of a function f defined by $df(x) = f'(x)\,dx = \frac{df}{dx}\,dx$. Another convenient notation is $f(x)\big|_{x=a}^{b}$ or simply $f(x)\big|_a^b$, which denotes the difference $f(b) - f(a)$.

Now the formula of integration by parts can be rewritten as

$$\int_a^b f(x)\,dg(x) = f(x)g(x)\bigg|_a^b - \int_a^b g(x)\,df(x)$$

for definite integrals and as

$$\int f\,dg = fg - \int g\,df$$

for indefinite integrals.

Examples

- $$\int \log x \, dx = x \log x - x + C \quad \text{on} \quad (0, \infty).$$

Integrating by parts, we obtain

$$\int \log x \, dx = x \log x - \int x \, d(\log x) = x \log x - \int x (\log x)' \, dx = x \log x - \int 1 \, dx = x \log x - x + C.$$

- $$\int_0^{\pi/2} x \sin x \, dx = 1.$$

Integrating by parts, we obtain

$$\int_0^{\pi/2} x \sin x \, dx = -x \cos x \Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) \, dx = \sin x \Big|_0^{\pi/2} = 1.$$

Examples

- $\displaystyle\int \log^3 x \, dx.$

We are going to integrate by parts several times:

$$\int \log^3 x \, dx = x \log^3 x - \int x \, d(\log^3 x) = x \log^3 x - \int x (\log^3 x)' \, dx$$

$$= x \log^3 x - \int 3 \log^2 x \, dx = x \log^3 x - 3x \log^2 x + \int x \, d(3 \log^2 x)$$

$$= x \log^3 x - 3x \log^2 x + \int 6 \log x \, dx$$

$$= x \log^3 x - 3x \log^2 x + 6x \log x - \int x \, d(6 \log x)$$

$$= x \log^3 x - 3x \log^2 x + 6x \log x - \int 6 \, dx$$

$$= x \log^3 x - 3x \log^2 x + 6x \log x - 6x + C.$$

Change of the variable in an integral

Theorem If ϕ is continuously differentiable on a closed, nondegenerate interval $[a, b]$ and f is continuous on $\phi([a, b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\,\phi'(x)\,dx = \int_a^b f(\phi(x))\,d\phi(x).$$

Remarks. • It is possible that $\phi(a) \geq \phi(b)$. To make sense of the integral in this case, we set

$$\int_c^d f(t)\,dt = -\int_d^c f(t)\,dt$$

if $c > d$. Also, we set the integral to be 0 if $c = d$.

• Substitution $t = \phi(x)$ is a proper change of the variable only if the function ϕ is strictly monotone. However the theorem holds even without this assumption.

Proof of the theorem: Let us define two functions:

$$F(u) = \int_{\phi(a)}^{u} f(t)\,dt, \quad u \in \phi([a,b]);$$

and

$$G(x) = \int_{a}^{x} f(\phi(s))\,\phi'(s)\,ds, \quad x \in [a,b].$$

It follows from the Fundamental Theorem of Calculus that $F'(u) = f(u)$ and $G'(x) = f(\phi(x))\,\phi'(x)$. By the Chain Rule,

$$(F \circ \phi)'(x) = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x) = G'(x).$$

Therefore $(F(\phi(x)) - G(x))' = 0$ for all $x \in [a,b]$. It follows that the function $F(\phi(x)) - G(x)$ is constant on $[a,b]$. In particular, $F(\phi(b)) - G(b) = F(\phi(a)) - G(a) = 0 - 0 = 0$.

Corollary Under assumptions of the theorem, if $\int f(t)\,dt = F(t) + C$ then $\int f(\phi(x))\,\phi'(x)\,dx = F(\phi(x)) + C$.

Examples

- $\displaystyle\int_0^{\pi} \sin^2(2x)\,dx.$

To integrate this function, we use a trigonometric formula $1-\cos(2\alpha)=2\sin^2\alpha$ and a new variable $u=4x$:

$$\int_0^{\pi} \sin^2(2x)\,dx = \int_0^{\pi} \frac{1-\cos(4x)}{2}\,dx$$

$$= \int_0^{\pi} \frac{1-\cos(4x)}{8}\,d(4x) = \int_0^{4\pi} \frac{1-\cos u}{8}\,du$$

$$= \left.\frac{u-\sin u}{8}\right|_{u=0}^{4\pi} = \frac{\pi}{2}.$$

Examples

- $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx.$

To integrate this function, we introduce a new variable $u = 1 - x^2$:

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx = -\frac{1}{2}\int_0^{1/2} \frac{(1-x^2)'}{\sqrt{1-x^2}}\, dx$$

$$= -\frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, d(1-x^2) = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\, du$$

$$= \int_{3/4}^{1} \frac{1}{2\sqrt{u}}\, du = \sqrt{u}\,\Big|_{u=3/4}^{1} = 1 - \frac{\sqrt{3}}{2}.$$

Examples

- $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx.$

To integrate this function, we use a substitution $x = 2\sin t$ (observe that x changes from 0 to 1 when t changes from 0 to $\pi/6$):

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \int_0^{\pi/6} \frac{1}{\sqrt{4-(2\sin t)^2}}\,d(2\sin t)$$

$$= \int_0^{\pi/6} \frac{(2\sin t)'}{\sqrt{4-4\sin^2 t}}\,dt = \int_0^{\pi/6} \frac{2\cos t}{\sqrt{4\cos^2 t}}\,dt$$

$$= \int_0^{\pi/6} \frac{2\cos t}{2\cos t}\,dt = \int_0^{\pi/6} 1\,dx = \frac{\pi}{6}.$$

Examples

- $\displaystyle\int \frac{\sqrt{1+\sqrt[4]{x}}}{2\sqrt{x}}\,dx.$

To find this integral, we change the variable twice. First

$$\int \frac{\sqrt{1+\sqrt[4]{x}}}{2\sqrt{x}}\,dx = \int \sqrt{1+\sqrt[4]{x}}\,(\sqrt{x})'\,dx = \int \sqrt{1+\sqrt{u}}\,du,$$

where $u=\sqrt{x}$. Secondly, we introduce a variable $w=\sqrt{1+\sqrt{u}}$. Then $u=(w^2-1)^2$ so that $du=\big((w^2-1)^2\big)'\,dw = 2(w^2-1)\cdot 2w\,dw = (4w^3-4w)\,dw$.
Consequently,

$$\int \sqrt{1+\sqrt{u}}\,du = \int w\,du = \int (4w^4-4w^2)\,dw$$

$$= \frac{4}{5}w^5 - \frac{4}{3}w^3 + C = \frac{4}{5}\big(1+x^{1/4}\big)^{5/2} - \frac{4}{3}\big(1+x^{1/4}\big)^{3/2} + C.$$