MATH 409–501 Spring 2020

Sample problems for Test 1

Any problem may be altered or replaced by a different one!

Problem 1. Prove the following version of the Archimedean property: for any positive real numbers x and y there exists a natural number n such that $nx > y$.

Problem 2. Prove that for any $n \in \mathbb{N}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Problem 3. Given a set X, let $\mathcal{P}(X)$ denote the set of all subsets of X. Prove that $\mathcal{P}(X)$ is not of the same cardinality as X.

Problem 4. Let $x_1 = a > 0$ and $x_{n+1} = 2\sqrt{x_n}$ for all $n \in \mathbb{N}$. Prove that the sequence $\{x_n\}$ is convergent and find its limit.

Problem 5. Suppose $\{r_n\}$ is a sequence that enumerates all rational numbers. Prove that every real number is a limit point of this sequence.

Problem 6. For each of the following series, determine whether the series converges and whether it converges absolutely:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$, (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+2^n\cos n}{n!}$, (iii) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\log n}$.