Math 412-501 October 20, 2006

Exam 2

Problem 1 (50 pts.) Solve the heat equation in a rectangle $0 < x < \pi$, $0 < y < \pi$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

subject to the initial condition

$$u(x, y, 0) = (\sin 2x + \sin 3x) \sin y$$

and the boundary conditions

$$u(0, y, t) = u(\pi, y, t) = 0, \qquad u(x, 0, t) = u(x, \pi, t) = 0.$$

Problem 2 (50 pts.) Solve Laplace's equation inside a quarter-circle $0 < r < 1$, $0 < \theta < \pi/2$ (in polar coordinates r, θ) subject to the boundary conditions

$$u(r, 0) = 0, \qquad u(r, \pi/2) = 0, \qquad |u(0, \theta)| < \infty, \qquad u(1, \theta) = f(\theta).$$

Bonus Problem 3 (40 pts.) Consider a regular Sturm-Liouville eigenvalue problem

$$\phi'' + \lambda\phi = 0, \qquad \phi'(0) = 0, \quad \phi'(1) + h\phi(1) = 0,$$

where h is a real constant.

(i) For what values of h is $\lambda = 0$ an eigenvalue?
(ii) For what values of h are all eigenvalues positive?
(iii) How many negative eigenvalues can this problem have?
(iv) Find an equation for positive eigenvalues.
(v) Find the asymptotics of λ_n as $n \to \infty$.