Math 412-501 November 17, 2006

Exam 3

Problem 1 (40 pts.) Solve the initial-boundary value problem for the wave equation in a semicircle (in polar coordinates r, θ)

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \qquad (0 < r < 1, \quad 0 < \theta < \pi),$$

$$u(r,\theta,0) = f(r)\sin 3\theta, \quad \frac{\partial u}{\partial t}(r,\theta,0) = 0 \qquad (0 < r < 1, \quad 0 < \theta < \pi),$$

$$u = 0 \text{ on the entire boundary.}$$

Problem 2 (25 pts.) It is known that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\beta x}\, dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-\beta^2/(4\alpha)}, \qquad \alpha > 0,\ \beta \in \mathbb{R}.$$

Let $f(x) = e^{-x^2/2}$, $x \in \mathbb{R}$.
(i) Find the Fourier transform of f.
(ii) Find the inverse Fourier transform of f.
(iii) Find an expression for the convolution $f * f$ that does not involve integrals.

Problem 3 (35 pts.) Solve the initial value problem for the heat equation on the infinite interval

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty, \quad t > 0),$$

$$u(x,0) = e^{-x^2/2}.$$

You cannot use Green's function unless you derive it.
Extra credit can be obtained when the solution will contain no integrals.

Bonus Problem 4 (35 pts.) Solve the initial-boundary value problem for the heat equation on the interval $[0,1]$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < 1, \quad t > 0),$$

$$u(x,0) = -\frac{1}{3}x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + 2\sin \pi x \qquad (0 < x < 1),$$

$$u(0,t) = t, \quad u(1,t) = 0.$$