Math 412-501 November 17, 2006

Exam 3: Solutions

Problem 1 (40 pts.) Solve the initial-boundary value problem for the wave equation in a semicircle (in polar coordinates r, θ)

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \qquad (0 < r < 1, \quad 0 < \theta < \pi),$$

$$u(r,\theta,0) = f(r)\sin 3\theta, \quad \frac{\partial u}{\partial t}(r,\theta,0) = 0 \qquad (0 < r < 1, \quad 0 < \theta < \pi),$$

$u = 0$ on the entire boundary.

Solution:

$$u(r,\theta,t) = \sum_{n=1}^{\infty} A_n \, J_3(j_{3,n} r)\, \sin 3\theta \, \cos(j_{3,n} t),$$

where $j_{3,n}$ is the nth positive zero of the Bessel function J_3 and

$$A_n = \frac{\displaystyle\int_0^1 f(r)\, J_3(j_{3,n} r)\, r\, dr}{\displaystyle\int_0^1 |J_3(j_{3,n} r)|^2\, r\, dr}.$$

Wave equation in polar coordinates:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We search for the solution of the problem as a superposition of normal modes, that is, solutions $u(r,\theta,t) = F(r)h(\theta)g(t)$ with separated variables of the wave equation satisfying the boundary condition. Substituting $u(r,\theta,t) = F(r)h(\theta)g(t)$ into the wave equation, we obtain

$$F(r)h(\theta)g''(t) = F''(r)h(\theta)g(t) + \frac{1}{r}F'(r)h(\theta)g(t) + \frac{1}{r^2}F(r)h''(\theta)g(t),$$

$$\frac{g''(t)}{g(t)} = \frac{F''(r)}{F(r)} + \frac{1}{r}\frac{F'(r)}{F(r)} + \frac{1}{r^2}\frac{h''(\theta)}{h(\theta)}.$$

Since the left-hand side does not depend on r and θ while the right-hand side does not depend on t, it follows that

$$\frac{g''(t)}{g(t)} = \frac{F''(r)}{F(r)} + \frac{1}{r}\frac{F'(r)}{F(r)} + \frac{1}{r^2}\frac{h''(\theta)}{h(\theta)} = -\lambda,$$

where λ is a constant. Then

$$r^2\frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} + \lambda r^2 = -\frac{h''(\theta)}{h(\theta)}.$$

The left-hand side of the latter equation does not depend on θ while its right-hand side does not depend on r. Therefore

$$r^2\frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} + \lambda r^2 = -\frac{h''(\theta)}{h(\theta)} = \mu,$$

where μ is a constant. Now all variables are separated:

$$r^2F''(r) + rF'(r) + (\lambda r^2 - \mu)F(r) = 0, \qquad h'' = -\mu h, \qquad g'' = -\lambda g.$$

Conversely, if functions F, h, and g are solutions of the above ODEs for the same values of λ and μ, then $u(r, \theta, t) = F(r)h(\theta)g(t)$ is a solution of the wave equation in polar coordinates.

The condition $u = 0$ on the entire boundary is equivalent to the following four conditions:

$$u(r, 0, t) = u(r, \pi, t) = 0, \qquad u(0, \theta, t) = u(1, \theta, t) = 0.$$

Substituting $u(r, \theta, t) = F(r)h(\theta)g(t)$ into these, we get

$$F(r)h(0)g(t) = F(r)h(\pi)g(t) = 0, \qquad F(0)h(\theta)g(t) = F(1)h(\theta)g(t) = 0.$$

It is no loss to assume that neither F nor h nor g is identically zero. Then the boundary conditions are satisfied if and only if $F(0) = F(1) = 0$, $h(0) = h(\pi) = 0$.

To determine h, we have an eigenvalue problem

$$h'' = -\mu h, \qquad h(0) = h(\pi) = 0.$$

This problem has eigenvalues $\mu_m = m^2$, $m = 1, 2, \ldots$. All eigenvalues are simple. The corresponding eigenfunctions are $h_m(\theta) = \sin m\theta$.

To determine F, we have another eigenvalue problem

$$r^2F''(r) + rF'(r) + (\lambda r^2 - \mu)F(r) = 0, \qquad F(0) = F(1) = 0.$$

We may assume that μ is one of the eigenvalues of the former eigenvalue problem, that is, $\mu = m^2$, where m is a positive integer. Also, we know that the latter eigenvalue problem is going to have only positive eigenvalues so we may assume that $\lambda > 0$.

Introduce a new coordinate $z = \sqrt{\lambda}\, r$. As a function of z, F satisfies Bessel's differential equation of order m:

$$z^2\frac{d^2F}{dz^2} + z\frac{dF}{dz} + (z^2 - m^2)F = 0.$$

Hence $F(z) = c_1J_m(z) + c_2Y_m(z)$, where c_1, c_2 are constants. Returning to the coordinate r, we obtain that $F(r) = c_1J_m(\sqrt{\lambda}\, r) + c_2Y_m(\sqrt{\lambda}\, r)$. The boundary condition $F(0) = 0$ holds if $c_2 = 0$. Then the boundary condition $F(1) = 0$ holds if $c_1J_m(\sqrt{\lambda}) = 0$. A nonzero solution exists if $J_m(\sqrt{\lambda}) = 0$. Therefore for any m we obtain a series of eigenvalues $\lambda_{m,1}, \lambda_{m,2}, \ldots$, where $\sqrt{\lambda_{m,n}}$ is the nth positive zero of the Bessel function J_m. All eigenvalues are simple. The corresponding eigenfunctions are $F_{m,n}(r) = J_m(\sqrt{\lambda_{m,n}}\, r)$.

The function g is to be determined from the equation $g'' = -\lambda g$. We may assume that λ is one of the above eigenvalues so that $\lambda > 0$. Then $g(t) = c_1 \cos(\sqrt{\lambda}\, t) + c_2 \sin(\sqrt{\lambda}\, t)$, where c_1, c_2 are constants.

Thus for any positive integers m and n we have the following normal modes:

$$u_{m,n}(r, \theta, t) = J_m(\sqrt{\lambda_{m,n}}\, r) \sin m\theta \cos(\sqrt{\lambda_{m,n}}\, t),$$

$$\tilde{u}_{m,n}(r, \theta, t) = J_m(\sqrt{\lambda_{m,n}}\, r) \sin m\theta \sin(\sqrt{\lambda_{m,n}}\, t).$$

Their superposition is a double series

$$u(r, \theta, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{m,n}}\, r)\, \sin m\theta \left(a_{m,n} \cos(\sqrt{\lambda_{m,n}}\, t) + b_{m,n} \sin(\sqrt{\lambda_{m,n}}\, t)\right),$$

where $a_{m,n}$ and $b_{m,n}$ are constants. The initial condition $\frac{\partial u}{\partial t}(r,\theta,0) = 0$ is satisfied if all $b_{m,n}$ are zero. The initial condition $u(r,\theta,0) = f(r)\sin 3\theta$ is satisfied if $a_{m,n} = 0$ for $m \neq 3$ and

$$\sum_{n=1}^{\infty} a_{3,n}\, J_3(\sqrt{\lambda_{3,n}}\, r)$$

is the Fourier-Bessel series of the function $f(r)$ on the interval $(0,1)$. That is, if

$$a_{3,n} = \frac{\displaystyle\int_0^1 f(r)\, J_3(\sqrt{\lambda_{3,n}}\, r)\, r\, dr}{\displaystyle\int_0^1 |J_3(\sqrt{\lambda_{3,n}}\, r)|^2\, r\, dr}.$$

Problem 2 (25 pts.) It is known that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\beta x}\, dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-\beta^2/(4\alpha)}, \qquad \alpha > 0,\ \beta \in \mathbb{R}.$$

Let $f(x) = e^{-x^2/2}$, $x \in \mathbb{R}$.
(i) Find the Fourier transform of f.
(ii) Find the inverse Fourier transform of f.
(iii) Find an expression for the convolution $f * f$ that does not involve integrals.

Solution: (i) $\mathcal{F}[f] = \dfrac{1}{\sqrt{2\pi}} f$; (ii) $\mathcal{F}^{-1}[f] = \sqrt{2\pi}\, f$; (iii) $(f * f)(x) = \sqrt{\pi}\, e^{-x^2/4}$.

For any $\omega \in \mathbb{R}$ we have that

$$\mathcal{F}[f](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2}\, e^{-i\omega x}\, dx = \frac{1}{2\pi}\sqrt{2\pi}\, e^{-\omega^2/2} = \frac{1}{\sqrt{2\pi}}\, e^{-\omega^2/2} = \frac{1}{\sqrt{2\pi}}\, f(\omega),$$

$$\mathcal{F}^{-1}[f](\omega) = \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{i\omega x}\, dx = \sqrt{2\pi}\, e^{-\omega^2/2} = \sqrt{2\pi}\, f(\omega).$$

By the convolution theorem, $\mathcal{F}[f * f] = 2\pi\, (\mathcal{F}[f])^2 = f^2$. Therefore

$$(f * f)(x) = \mathcal{F}^{-1}[f^2](x) = \int_{-\infty}^{\infty} (f(\omega))^2\, e^{i\omega x}\, d\omega = \int_{-\infty}^{\infty} e^{-\omega^2} e^{i\omega x}\, d\omega = \sqrt{\pi}\, e^{-x^2/4}.$$

Problem 3 (35 pts.) Solve the initial value problem for the heat equation on the infinite interval

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty, \quad t > 0),$$

$$u(x,0) = e^{-x^2/2}.$$

You cannot use Green's function unless you derive it.
Extra credit can be obtained when the solution will contain no integrals.

Solution: $u(x,t) = \dfrac{1}{\sqrt{2t+1}}\, e^{-x^2/(4t+2)}$.

The function $f(x) = e^{-x^2/2}$ is smooth and rapidly decaying as $x \to \infty$. This suggests that the solution $u(x,t)$ will have the same properties.

Apply the Fourier transform (relative to x) to both sides of the equation:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right].$$

Let U denote the Fourier transform of the solution u,

$$U(\omega,t) = \mathcal{F}[u(\cdot,t)](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx.$$

Then

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial U}{\partial t}, \qquad \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (i\omega)^2 U(\omega,t) = -\omega^2 U(\omega,t).$$

Therefore

$$\frac{\partial U}{\partial t} = -\omega^2 U(\omega,t).$$

For any $\omega \in \mathbb{R}$ this is an ordinary differential equation in variable t. The general solution is $U(\omega,t) = ce^{-\omega^2 t}$, where c is a constant. Note that c depends on ω, $c = c(\omega)$.

The initial condition $u(x,0) = f(x) = e^{-x^2/2}$ implies that $U(\omega,0) = \hat{f}(\omega)$. Hence $c(\omega) = \hat{f}(\omega)$ for all $\omega \in \mathbb{R}$ and $U(\omega,t) = \hat{f}(\omega)e^{-\omega^2 t}$. As shown in the solution of Problem 2, $\hat{f}(\omega) = (2\pi)^{-1/2}e^{-\omega^2/2}$. Then

$$U(\omega,t) = \frac{1}{\sqrt{2\pi}}\, e^{-\omega^2/2}\, e^{-\omega^2 t} = \frac{1}{\sqrt{2\pi}}\, e^{-(t+1/2)\omega^2}.$$

It remains to apply the inverse Fourier transform:

$$u(x,t) = \mathcal{F}^{-1}[U(\cdot,t)](x) = \int_{-\infty}^{\infty} U(\omega,t)e^{i\omega x}\,d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(t+1/2)\omega^2}e^{i\omega x}\,d\omega$$
$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{t+1/2}}\, e^{-x^2/(4t+2)} = \frac{1}{\sqrt{2t+1}}\, e^{-x^2/(4t+2)}.$$

Bonus Problem 4 (35 pts.) Solve the initial-boundary value problem for the heat equation on the interval $[0,1]$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < 1, \quad t > 0),$$
$$u(x,0) = -\frac{1}{3}x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + 2\sin \pi x \qquad (0 < x < 1),$$
$$u(0,t) = t, \quad u(1,t) = 0.$$

Solution: $u(x,t) = t(1-x) - \dfrac{1}{3}x + \dfrac{1}{2}x^2 - \dfrac{1}{6}x^3 + 2e^{-\pi^2 t}\sin \pi x.$

The boundary conditions are not homogeneous. The function $u_0(x,t) = t(1-x)$ satisfies them. Let $u(x,t)$ be the solution of the problem. Then the function $w(x,t) = u(x,t) - u_0(x,t)$ satisfies homogeneous boundary conditions $w(0,t) = w(1,t) = 0$. Since $u = w + u_0$, we obtain

$$\frac{\partial}{\partial t}(w+u_0) = \frac{\partial^2}{\partial x^2}(w+u_0),$$

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = -\frac{\partial u_0}{\partial t} + \frac{\partial^2 u_0}{\partial x^2} = x - 1.$$

Also, $w(x,0) = u(x,0) - u_0(x,0) = u(x,0)$. Therefore w is the solution of the following initial-boundary value problem:

$$\begin{aligned}
&\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + x - 1 \qquad (0 < x < 1, \quad t > 0),\\
&w(x,0) = -\frac{1}{3}x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + 2\sin \pi x \qquad (0 < x < 1),\\
&w(0,t) = w(1,t) = 0.
\end{aligned}$$

The solution can be represented in the form $w(x,t) = w_0(x) + v(x,t)$, where w_0 is the steady-state solution of the above equation that satisfies the boundary conditions:

$$\frac{d^2 w_0}{dx^2} + x - 1 = 0, \qquad w_0(0) = w_0(1) = 0,$$

while v is the solution of an initial-boundary value problem for the homogeneous heat equation:

$$\begin{aligned}
&\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \qquad (0 < x < 1, \quad t > 0),\\
&v(x,0) = -w_0(x) - \frac{1}{3}x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + 2\sin \pi x \qquad (0 < x < 1),\\
&v(0,t) = v(1,t) = 0.
\end{aligned}$$

First we have to find w_0. Since $w_0''(x) = 1 - x$, it follows that $w_0(x) = \frac{1}{2}x^2 - \frac{1}{6}x^3 + c_1 x + c_2$, where c_1, c_2 are constants. The boundary conditions $w_0(0) = w_0(1) = 0$ hold when $c_1 = -\frac{1}{3}$, $c_2 = 0$. Thus

$$w_0(x) = -\frac{1}{3}x + \frac{1}{2}x^2 - \frac{1}{6}x^3.$$

Now we know the initial condition that v should satisfy: $v(x,0) = 2\sin \pi x$. Observe that $\phi(x) = \sin \pi x$ is an eigenfunction of the problem

$$\phi'' = -\lambda \phi, \qquad \phi(0) = \phi(1) = 0.$$

The corresponding eigenvalue is $\lambda = \pi^2$. Hence v is a solution with separated variables. It is easy to obtain that $v(x,t) = 2e^{-\pi^2 t} \sin \pi x$.

Finally, $u(x,t) = u_0(x,t) + w_0(x) + v(x,t) = t(1-x) - \frac{1}{3}x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + 2e^{-\pi^2 t} \sin \pi x$.