Math 412-501 Fall 2006

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Some possibly useful information

- Parseval's equality for the complex form of the Fourier series on $(-\pi, \pi)$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \Longrightarrow \quad \int_{-\pi}^{\pi} |f(x)|^2\, dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2.$$

- Fourier sine and cosine transforms of the second derivative:

$$S[f''](\omega) = \frac{2}{\pi}\, f(0)\,\omega - \omega^2 S[f](\omega), \qquad C[f''](\omega) = -\frac{2}{\pi}\, f'(0) - \omega^2 C[f](\omega).$$

- Laplace's operator in polar coordinates r, θ:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

- Any nonzero solution of a regular Sturm-Liouville equation

$$(p\phi')' + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b)$$

satisfies the Rayleigh quotient relation

$$\lambda = \frac{-p\phi\phi'\Big|_a^b + \displaystyle\int_a^b \big(p(\phi')^2 - q\phi^2\big)\, dx}{\displaystyle\int_a^b \phi^2\sigma\, dx}.$$

- Some table integrals:

$$\int x^2 e^{iax}\, dx = \left(\frac{x^2}{ia} + \frac{2x}{a^2} - \frac{2}{ia^3}\right) e^{iax} + C, \quad a \neq 0;$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\beta x}\, dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-\beta^2/(4\alpha)}, \quad \alpha > 0,\ \beta \in \mathbb{R};$$

$$\int_{-\infty}^{\infty} e^{-\alpha|x|} e^{i\beta x}\, dx = \frac{2\alpha}{\alpha^2 + \beta^2}, \quad \alpha > 0,\ \beta \in \mathbb{R}.$$

Problem 1 Let $f(x) = x^2$.
(i) Find the Fourier series (complex form) of $f(x)$ on the interval $(-\pi, \pi)$.
(ii) Rewrite the Fourier series of $f(x)$ in the real form.
(iii) Sketch the function to which the Fourier series converges.
(iv) Use Parseval's equality to evaluate $\sum_{n=1}^{\infty} n^{-4}$.

Problem 2 Solve Laplace's equation in a disk,

$$\nabla^2 u = 0 \quad (0 \leq r < a), \qquad u(a, \theta) = f(\theta).$$

Problem 3 Find Green's function for the boundary value problem

$$\frac{d^2 u}{dx^2} - u = f(x) \quad (0 < x < 1), \qquad u'(0) = u'(1) = 0.$$

Problem 4 Solve the initial-boundary value problem for the heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, \quad t > 0),$$

$$u(x, 0) = f(x) \qquad (0 < x < \pi),$$

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(\pi, t) + 2u(\pi, t) = 0.$$

In the process you will discover a sequence of eigenfunctions and eigenvalues, which you should name $\phi_n(x)$ and λ_n. Describe the λ_n qualitatively (e.g., find an equation for them) but do not expect to find their exact numerical values. Also, do not bother to evaluate normalization integrals for ϕ_n.

Problem 5 By the method of your choice, solve the wave equation on the half-line

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty, \ -\infty < t < \infty)$$

subject to

$$u(0, t) = 0, \qquad u(x, 0) = f(x), \qquad \frac{\partial u}{\partial t}(x, 0) = g(x).$$

Bonus Problem 6 Solve Problem 5 by a distinctly different method.

Bonus Problem 7 Find a Green function implementing the solution of Problem 2.