Homework assignment #2
(due Friday, September 15)

Problem 1. Show that the equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(u, x, t)$$

is linear if $Q(u, x, t) = \alpha(x, t)u + \beta(x, t)$ and in addition homogeneous if $\beta(x, t) = 0$.

Problem 2. Show that a linear equation is homogeneous if and only if 0 is a solution.

Problem 3. Consider the following equation:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + u^2.$$

(i) Find a nonzero steady-state (independent of t) solution u_0 in the half-plane $x > 0$;
(ii) show that $2u_0$ is not a solution;
(iii) use u_0 to show that the equation is not linear.

Problem 4. Using separation of variables, find a nonzero solution of the equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} - u \qquad (k = \text{const} > 0).$$

Problem 5. Determine the eigenvalues λ of the following eigenvalue problem:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0.$$

Analyze three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$. You may assume that the eigenvalues are real.

Problem 6. Solve the initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0$$

with the following initial and boundary conditions:

(i) $u(x, 0) = 6\sin\frac{9\pi x}{L}$, $\quad u(0, t) = u(L, t) = 0$;

(ii) $u(x, 0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$, $\quad u(0, t) = u(L, t) = 0$;

(iii) $u(x, 0) = 6 + 4\cos\frac{3\pi x}{L}$, $\quad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$;

(iv) $u(x, 0) = -3\cos\frac{8\pi x}{L}$, $\quad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$.

Problem 7. Show that all solutions of Problem 6 uniformly approach steady-state solutions as $t \to \infty$.