Solutions for homework assignment #4

Problem 1. Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions:

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = 0, \qquad u(x,H) = f(x).$$

Solution:

$$u(x,y) = b_0 \frac{y}{H} + \sum\nolimits_{n=1}^{\infty} b_n \left(\sinh \frac{n\pi H}{L}\right)^{-1} \sinh \frac{n\pi y}{L} \cos \frac{n\pi x}{L},$$

where

$$b_0 + \sum\nolimits_{n=1}^{\infty} b_n \cos \frac{n\pi x}{L}$$

is the Fourier cosine series of the function $f(x)$ on $[0,L]$, that is,

$$b_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad b_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \dots$$

Detailed solution: We search for the solution of the boundary value problem as a superposition of solutions $u(x,y) = \phi(x)h(y)$ with separated variables of Laplace's equation that satisfy the three homogeneous boundary conditions.

Substituting $u(x,y) = \phi(x)h(y)$ into Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

we obtain

$$\phi''(x)h(y) + \phi(x)h''(y) = 0,$$
$$\frac{\phi''(x)}{\phi(x)} = -\frac{h''(y)}{h(y)}.$$

Since the left-hand side does not depend on y while the right-hand side does not depend on x, it follows that

$$\frac{\phi''(x)}{\phi(x)} = -\frac{h''(y)}{h(y)} = -\lambda,$$

where λ is a constant. Then

$$\phi'' = -\lambda\phi, \qquad h'' = \lambda h.$$

Conversely, if functions ϕ and h are solutions of the above ODEs for the same value of λ, then $u(x,y) = \phi(x)h(y)$ is a solution of Laplace's equation.

Substituting $u(x,y) = \phi(x)h(y)$ into the homogeneous boundary conditions, we get

$$\phi'(0)h(y) = 0, \quad \phi'(L)h(y) = 0, \quad \phi(x)h(0) = 0.$$

It is no loss to assume that neither ϕ nor h is identically zero. Then the boundary conditions are satisfied if and only if $\phi'(0) = \phi'(L) = 0$, $h(0) = 0$.

To determine ϕ, we have an eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi'(0) = \phi'(L) = 0.$$

This problem has eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \ldots$. The corresponding eigenfunctions are $\phi_0 = 1$ and $\phi_n(x) = \cos\frac{n\pi x}{L}$, $n = 1, 2, \ldots$.

The function h is to be determined from the equation $h'' = \lambda h$ and the boundary condition $h(0) = 0$. We may assume that λ is one of the above eigenvalues so that $\lambda \geq 0$. If $\lambda = 0$ then the general solution of the equation is $h(y) = c_1 + c_2 y$, where c_1, c_2 are constants. If $\lambda > 0$ then the general solution is $h(y) = c_1 \cosh \mu y + c_2 \sinh \mu y$, where $\mu = \sqrt{\lambda}$ and c_1, c_2 are constants. In both cases, the boundary condition $h(0) = 0$ holds if $c_1 = 0$.

Thus we obtain the following solutions of Laplace's equation satisfying the three homogeneous boundary conditions:

$$u_0(x,y) = y, \quad u_n(x,y) = \sinh\frac{n\pi y}{L}\cos\frac{n\pi x}{L}, \ n = 1, 2, \ldots$$

A superposition of these solutions is a series

$$u(x,y) = c_0 y + \sum\nolimits_{n=1}^{\infty} c_n \sinh\frac{n\pi y}{L}\cos\frac{n\pi x}{L},$$

where $c_0, c_1, c_2, \ldots$ are constants. Substituting the series into the boundary condition $u(x,H) = f(x)$, we get

$$f(x) = c_0 H + \sum\nolimits_{n=1}^{\infty} c_n \sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L}.$$

The right-hand side is a Fourier cosine series on the interval $[0, L]$. Therefore the boundary condition is satisfied if the right-hand side coincides with the Fourier cosine series

$$b_0 + \sum\nolimits_{n=1}^{\infty} b_n \cos\frac{n\pi x}{L}$$

of the function $f(x)$ on $[0, L]$. Hence

$$c_0 = \frac{b_0}{H}, \qquad c_n = \frac{b_n}{\sinh\frac{n\pi H}{L}}, \quad n = 1, 2, \ldots,$$

where

$$b_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad b_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots$$

Problem 2. Solve Laplace's equation inside a semicircle of radius a ($0 < r < a$, $0 < \theta < \pi$) subject to the boundary conditions: $u = 0$ on the diameter and $u(a, \theta) = g(\theta)$.

Solution:

$$u(r,\theta) = \sum\nolimits_{n=1}^{\infty} b_n \left(\frac{r}{a}\right)^n \sin n\theta,$$

where

$$\sum\nolimits_{n=1}^{\infty} b_n \sin n\theta$$

is the Fourier sine series of the function $g(\theta)$ on $[0, \pi]$, that is,

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin n\theta\,d\theta, \quad n = 1, 2, \ldots$$

Detailed solution: Laplace's equation in polar coordinates (r, θ):

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

The boundary condition $u = 0$ on the diameter gives rise to three boundary conditions in polar coordinates:

$$u(r, 0) = u(r, \pi) = 0 \qquad (0 < r < a),$$
$$u(0, \theta) = 0 \qquad (0 < \theta < \pi)$$

(the latter condition means that $u = 0$ at the origin).

We search for the solution of the boundary value problem as a superposition of solutions $u(r, \theta) = h(r)\phi(\theta)$ with separated variables of Laplace's equation that satisfy the above homogeneous boundary conditions.

Substituting $u(r, \theta) = h(r)\phi(\theta)$ into Laplace's equation, we obtain

$$h''(r)\phi(\theta) + \frac{1}{r} h'(r)\phi(\theta) + \frac{1}{r^2} h(r)\phi''(\theta) = 0,$$
$$\frac{r^2 h''(r) + rh'(r)}{h(r)} = -\frac{\phi''(\theta)}{\phi(\theta)}.$$

Since the left-hand side does not depend on θ while the right-hand side does not depend on r, it follows that

$$\frac{r^2 h''(r) + rh'(r)}{h(r)} = -\frac{\phi''(\theta)}{\phi(\theta)} = \lambda,$$

where λ is a constant. Then

$$r^2 h''(r) + rh'(r) = \lambda h(r), \qquad \phi'' = -\lambda\phi.$$

Conversely, if functions h and ϕ are solutions of the above ODEs for the same value of λ, then $u(r, \theta) = h(r)\phi(\theta)$ is a solution of Laplace's equation in polar coordinates.

Substituting $u(r, \theta) = h(r)\phi(\theta)$ into the homogeneous boundary conditions, we get

$$h(r)\phi(0) = 0, \quad h(r)\phi(\pi) = 0, \quad h(0)\phi(\theta) = 0.$$

It is no loss to assume that neither h nor ϕ is identically zero. Then the boundary conditions are satisfied if and only if $\phi(0) = \phi(\pi) = 0$, $h(0) = 0$.

To determine ϕ, we have an eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = \phi(\pi) = 0.$$

This problem has eigenvalues $\lambda_n = n^2$, $n = 1, 2, \dots$. The corresponding eigenfunctions are $\phi_n(\theta) = \sin n\theta$, $n = 1, 2, \dots$.

The function h is to be determined from the equation $r^2 h'' + rh' = \lambda h$ and the boundary condition $h(0) = 0$. We may assume that λ is one of the above eigenvalues so that $\lambda > 0$. Then the general solution of the equation is $h(r) = c_1 r^\mu + c_2 r^{-\mu}$, where $\mu = \sqrt{\lambda}$ and c_1, c_2 are constants. The boundary condition $h(0) = 0$ holds if $c_2 = 0$.

Thus we obtain the following solutions of Laplace's equation satisfying the three homogeneous boundary conditions:

$$u_n(r, \theta) = r^n \sin n\theta, \quad n = 1, 2, \dots$$

A superposition of these solutions is a series

$$u(r, \theta) = \sum_{n=1}^{\infty} c_n r^n \sin n\theta,$$

where $c_1, c_2, \dots$ are constants. Substituting the series into the boundary condition $u(a, \theta) = g(\theta)$, we get

$$g(\theta) = \sum_{n=1}^{\infty} c_n a^n \sin n\theta.$$

The right-hand side is a Fourier sine series on the interval $[0, \pi]$. Therefore the boundary condition is satisfied if the right-hand side coincides with the Fourier sine series

$$\sum\nolimits_{n=1}^{\infty} b_n \sin n\theta$$

of the function $g(\theta)$ on $[0, \pi]$. Hence

$$c_n = b_n a^{-n}, \quad n = 1, 2, \dots,$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} g(\theta) \sin n\theta \, d\theta, \quad n = 1, 2, \dots$$

Problem 3. Solve Laplace's equation inside a 90° sector of a circular annulus ($a < r < b$, $0 < \theta < \pi/2$) subject to the boundary conditions:

$$u(r, 0) = 0, \qquad u(r, \pi/2) = 0, \qquad u(a, \theta) = 0, \qquad u(b, \theta) = f(\theta).$$

Solution:

$$u(r, \theta) = \sum\nolimits_{n=1}^{\infty} b_n \frac{(r/a)^{2n} - (a/r)^{2n}}{(b/a)^{2n} - (a/b)^{2n}} \sin 2n\theta,$$

where

$$\sum\nolimits_{n=1}^{\infty} b_n \sin 2n\theta$$

is the Fourier sine series of the function $f(\theta)$ on $[0, \pi/2]$, that is,

$$b_n = \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \sin 2n\theta \, d\theta, \quad n = 1, 2, \dots$$

Detailed solution: We search for the solution of the boundary value problem as a superposition of solutions $u(r, \theta) = h(r)\phi(\theta)$ with separated variables of Laplace's equation that satisfy the three homogeneous boundary conditions.

As shown in the solution of Problem 2, $u(r, \theta) = h(r)\phi(\theta)$ is a solution of Laplace's equation in polar coordinates if functions h and ϕ are solutions of the equations

$$r^2 h''(r) + r h'(r) = \lambda h(r), \qquad \phi'' = -\lambda \phi$$

for the same constant λ.

Substituting $u(r, \theta) = h(r)\phi(\theta)$ into the homogeneous boundary conditions, we get

$$h(r)\phi(0) = 0, \quad h(r)\phi(\pi/2) = 0, \quad h(a)\phi(\theta) = 0.$$

It is no loss to assume that neither h nor ϕ is identically zero. Then the boundary conditions are satisfied if and only if $\phi(0) = \phi(\pi/2) = 0$, $h(a) = 0$.

To determine ϕ, we have an eigenvalue problem

$$\phi'' = -\lambda \phi, \qquad \phi(0) = \phi(\pi/2) = 0.$$

This problem has eigenvalues $\lambda_n = (2n)^2$, $n = 1, 2, \dots$. The corresponding eigenfunctions are $\phi_n(\theta) = \sin 2n\theta$, $n = 1, 2, \dots$.

The function h is to be determined from the equation $r^2h''+rh' = \lambda h$ and the boundary condition $h(a) = 0$. We may assume that λ is one of the above eigenvalues so that $\lambda > 0$. Then the general solution of the equation is $h(r) = c_1r^{\mu}+c_2r^{-\mu}$, where $\mu = \sqrt{\lambda}$ and c_1, c_2 are constants. The boundary condition $h(a) = 0$ holds if $c_1a^{\mu} + c_2a^{-\mu} = 0$, which implies that $h(r) = c_0((r/a)^{\mu} - (r/a)^{-\mu})$, where c_0 is a constant.

Thus we obtain the following solutions of Laplace's equation satisfying the three homogeneous boundary conditions:

$$u_n(r,\theta) = \left(\left(\frac{r}{a}\right)^{2n} - \left(\frac{a}{r}\right)^{2n}\right)\sin 2n\theta, \quad n = 1,2,\dots$$

A superposition of these solutions is a series

$$u(r,\theta) = \sum\nolimits_{n=1}^{\infty} c_n\left(\left(\frac{r}{a}\right)^{2n} - \left(\frac{a}{r}\right)^{2n}\right)\sin 2n\theta,$$

where $c_1, c_2, \dots$ are constants. Substituting the series into the boundary condition $u(b,\theta) = f(\theta)$, we get

$$f(\theta) = \sum\nolimits_{n=1}^{\infty} c_n\left(\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right)\sin 2n\theta.$$

The right-hand side is a Fourier sine series on the interval $[0,\pi/2]$. Therefore the boundary condition is satisfied if the right-hand side coincides with the Fourier sine series

$$\sum\nolimits_{n=1}^{\infty} b_n \sin 2n\theta$$

of the function $f(\theta)$ on $[0,\pi/2]$. Hence

$$c_n = \frac{b_n}{(b/a)^{2n} - (a/b)^{2n}}, \quad n = 1,2,\dots,$$

where

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin 2n\theta\,d\theta, \quad n = 1,2,\dots$$

Problem 4. Consider the heat equation in a two-dimensional rectangular region, $0 < x < L$, $0 < y < H$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial condition $u(x,y,0) = f(x,y)$.

Solve the initial-boundary value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are:

$$\frac{\partial u}{\partial x}(0,y,t) = 0, \qquad \frac{\partial u}{\partial x}(L,y,t) = 0, \qquad \frac{\partial u}{\partial y}(x,0,t) = 0, \qquad \frac{\partial u}{\partial y}(x,H,t) = 0.$$

Solution:

$$u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\exp\Big(-\big((n\pi/L)^2 + (m\pi/H)^2\big)\,kt\Big)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H},$$

where

$$c_{00} = \frac{1}{LH}\int_0^L\int_0^H f(x,y)\,dx\,dy,$$

$$c_{n0} = \frac{2}{LH}\int_0^L\int_0^H f(x,y)\cos\frac{n\pi x}{L}\,dx\,dy, \quad n \geq 1,$$

$$c_{0m} = \frac{2}{LH}\int_0^L\int_0^H f(x,y)\cos\frac{m\pi y}{H}\,dx\,dy, \quad m \geq 1,$$

$$c_{nm} = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\,dx\,dy, \quad n,m \geq 1.$$

As $t \to \infty$, the temperature uniformly approaches the constant c_{00}, the mean value of $f(x,y)$ over the rectangle.

Detailed solution: We search for the solution of the initial-boundary value problem as a superposition of solutions $u(x,y,t) = \phi(x)h(y)G(t)$ with separated variables of the heat equation that satisfy the boundary conditions.

Substituting $u(x,y,t) = \phi(x)h(y)G(t)$ into the heat equation, we obtain

$$\phi(x)h(y)G'(t) = k\Big(\phi''(x)h(y)G(t) + \phi(x)h''(y)G(t)\Big),$$

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} + \frac{h''(y)}{h(y)}.$$

Since any of the expressions $\frac{G'(t)}{kG(t)}$, $\frac{\phi''(x)}{\phi(x)}$, and $\frac{h''(y)}{h(y)}$ depend on one of the variables x, y, t and does not depend on the other two, it follows that each of these expressions is constant. Hence

$$\frac{\phi''(x)}{\phi(x)} = -\lambda, \qquad \frac{h''(y)}{h(y)} = -\mu, \qquad \frac{G'(t)}{kG(t)} = -(\lambda+\mu),$$

where λ and μ are constants. Then

$$\phi'' = -\lambda\phi, \qquad h'' = -\mu h, \qquad G' = -(\lambda+\mu)kG.$$

Conversely, if functions ϕ, h, and G are solutions of the above ODEs for the same values of λ and μ, then $u(x,y,t) = \phi(x)h(y)G(t)$ is a solution of the heat equation.

Substituting $u(x,y,t) = \phi(x)h(y)G(t)$ into the boundary conditions, we get

$$\phi'(0)h(y)G(t) = \phi'(L)h(y)G(t) = 0, \qquad \phi(x)h'(0)G(t) = \phi(x)h'(H)G(t) = 0.$$

It is no loss to assume that neither ϕ nor h nor G is identically zero. Then the boundary conditions are satisfied if and only if $\phi'(0) = \phi'(L) = 0$, $h'(0) = h'(H) = 0$.

To determine ϕ, we have an eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi'(0) = \phi'(L) = 0.$$

This problem has eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \dots$. The corresponding eigenfunctions are $\phi_0 = 1$ and $\phi_n(x) = \cos\frac{n\pi x}{L}$, $n = 1, 2, \dots$.

To determine h, we have another eigenvalue problem

$$h'' = -\mu h, \qquad h'(0) = h'(H) = 0.$$

This problem has eigenvalues $\mu_m = \left(\dfrac{m\pi}{H}\right)^2$, $m = 0, 1, 2, \ldots$. The corresponding eigenfunctions are $\psi_0 = 1$ and $\psi_m(y) = \cos\dfrac{m\pi y}{H}$, $m = 1, 2, \ldots$.

The function G is to be determined from the equation $G' = -(\lambda + \mu)kG$. The general solution of this equation is $G(t) = c_0 e^{-(\lambda+\mu)kt}$, where c_0 is a constant.

Thus we obtain the following solutions of the heat equation satisfying the boundary conditions:

$$\begin{aligned} u_{nm}(x, y, t) &= e^{-(\lambda_n+\mu_m)kt}\phi_n(x)\psi_m(y) \\ &= \exp\Big(-\big((n\pi/L)^2 + (m\pi/H)^2\big)\,kt\Big)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}, \qquad n, m = 0, 1, 2, \ldots \end{aligned}$$

A superposition of these solutions is a double series

$$u(x, y, t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\exp\Big(-\big((n\pi/L)^2 + (m\pi/H)^2\big)\,kt\Big)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H},$$

where c_{nm} are constants. Substituting the series into the initial condition $u(x, y, 0) = f(x, y)$, we get

$$f(x, y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\phi_n(x)\psi_m(y).$$

To determine the coefficients c_{nm}, we multiply both sides by $\phi_N(x)\psi_M(y)$ $(N, M \geq 0)$ and integrate over the rectangle $0 \leq x \leq L$, $0 \leq y \leq H$. We assume that the series may be integrated term-by-term:

$$\begin{aligned} \int_0^L\!\!\int_0^H f(x, y)\phi_N(x)\psi_M(y)\,dx\,dy &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\int_0^L\!\!\int_0^H \phi_N(x)\psi_M(y)\phi_n(x)\psi_m(y)\,dx\,dy \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\int_0^L \phi_N(x)\phi_n(x)\,dx\int_0^H \psi_M(y)\psi_m(y)\,dy. \end{aligned}$$

Using the orthogonality relations

$$\int_0^L \phi_N(x)\phi_n(x)\,dx = 0, \quad N \neq n,$$

$$\int_0^H \psi_M(y)\psi_m(y)\,dy = 0, \quad M \neq m,$$

we obtain

$$\int_0^L\!\!\int_0^H f(x, y)\phi_N(x)\psi_M(y)\,dx\,dy = c_{NM}\int_0^L \phi_N^2(x)\,dx\int_0^H \psi_M^2(y)\,dy.$$

It remains to recall that

$$\int_0^L \phi_0^2(x)\,dx = L, \qquad \int_0^L \phi_N^2(x)\,dx = \frac{L}{2}, \quad N \geq 1,$$

and, similarly,

$$\int_0^H \psi_0^2(x)\,dx = H, \qquad \int_0^H \psi_M^2(x)\,dx = \frac{H}{2}, \quad M \geq 1.$$

In the double series expansion of $u(x, y, t)$, each term contains an exponential factor $e^{-(\lambda_n+\mu_m)kt}$, which is decaying as $t \to \infty$ except for the case $n = m = 0$ when this factor is equal to 1. It follows that, as $t \to \infty$, the solution $u(x, y, t)$ uniformly converges to the constant c_{00}:

$$\lim_{t\to\infty} u(x, y, t) = c_{00} = \frac{1}{LH}\int_0^L\!\!\int_0^H f(x, y)\,dx\,dy.$$

Problem 5. Consider the wave equation for a vibrating rectangular membrane ($0 < x < L$, $0 < y < H$)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the initial conditions $u(x, y, 0) = 0$ and $\dfrac{\partial u}{\partial t}(x, y, 0) = f(x, y)$.

Solve the initial-boundary value problem if

$$\frac{\partial u}{\partial x}(0, y, t) = 0, \qquad \frac{\partial u}{\partial x}(L, y, t) = 0, \qquad \frac{\partial u}{\partial y}(x, 0, t) = 0, \qquad \frac{\partial u}{\partial y}(x, H, t) = 0.$$

Solution:

$$u(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_{nm} \frac{\sin(\sqrt{(n\pi/L)^2 + (m\pi/H)^2}\, ct)}{\sqrt{(n\pi/L)^2 + (m\pi/H)^2}\, c} \cos \frac{n\pi x}{L} \cos \frac{m\pi y}{H},$$

where b_{nm} are coefficients of the expansion

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_{nm} \cos \frac{n\pi x}{L} \cos \frac{m\pi y}{H}.$$

The formulas for b_{nm} are obtained in the solution of Problem 4.

Detailed solution: We search for the solution of the initial-boundary value problem as a superposition of solutions $u(x, y, t) = \phi(x)h(y)G(t)$ with separated variables of the wave equation that satisfy the boundary conditions.

Substituting $u(x, y, t) = \phi(x)h(y)G(t)$ into the wave equation, we obtain

$$\phi(x)h(y)G''(t) = c^2 \Big(\phi''(x)h(y)G(t) + \phi(x)h''(y)G(t) \Big),$$

$$\frac{G''(t)}{c^2 G(t)} = \frac{\phi''(x)}{\phi(x)} + \frac{h''(y)}{h(y)}.$$

Since any of the expressions $\dfrac{G''(t)}{c^2 G(t)}$, $\dfrac{\phi''(x)}{\phi(x)}$, and $\dfrac{h''(y)}{h(y)}$ depend on one of the variables x, y, t and does not depend on the other two, it follows that each of these expressions is constant. Hence

$$\frac{\phi''(x)}{\phi(x)} = -\lambda, \qquad \frac{h''(y)}{h(y)} = -\mu, \qquad \frac{G''(t)}{c^2 G(t)} = -(\lambda + \mu),$$

where λ and μ are constants. Then

$$\phi'' = -\lambda \phi, \qquad h'' = -\mu h, \qquad G'' = -(\lambda + \mu)c^2 G.$$

Conversely, if functions ϕ, h, and G are solutions of the above ODEs for the same values of λ and μ, then $u(x, y, t) = \phi(x)h(y)G(t)$ is a solution of the wave equation.

Substituting $u(x, y, t) = \phi(x)h(y)G(t)$ into the boundary conditions, we get

$$\phi'(0)h(y)G(t) = \phi'(L)h(y)G(t) = 0, \qquad \phi(x)h'(0)G(t) = \phi(x)h'(H)G(t) = 0.$$

It is no loss to assume that neither ϕ nor h nor G is identically zero. Then the boundary conditions are satisfied if and only if $\phi'(0) = \phi'(L) = 0$, $h'(0) = h'(H) = 0$.

To determine ϕ, we have an eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi'(0) = \phi'(L) = 0.$$

This problem has eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \ldots$. The corresponding eigenfunctions are $\phi_0 = 1$ and $\phi_n(x) = \cos\frac{n\pi x}{L}$, $n = 1, 2, \ldots$.

To determine h, we have another eigenvalue problem

$$h'' = -\mu h, \qquad h'(0) = h'(H) = 0.$$

This problem has eigenvalues $\mu_m = \left(\frac{m\pi}{H}\right)^2$, $m = 0, 1, 2, \ldots$. The corresponding eigenfunctions are $\psi_0 = 1$ and $\psi_m(y) = \cos\frac{m\pi y}{H}$, $m = 1, 2, \ldots$.

The function G is to be determined from the equation $G'' = -(\lambda + \mu)c^2 G$. We may assume that λ and μ are eigenvalues of the above eigenvalue problems so that $\lambda, \mu \geq 0$. If $\lambda = \mu = 0$ then the general solution of the equation is $G(t) = C_0 + D_0 t$, where C_0, D_0 are constants. If $\lambda + \mu > 0$ then the general solution of the equation is

$$G(t) = C_0 \cos(\sqrt{\lambda + \mu}\, ct) + D_0 \sin(\sqrt{\lambda + \mu}\, ct),$$

where C_0, D_0 are constants.

Thus for any $n, m \geq 0$ we have the following solutions of the wave equation satisfying the boundary conditions:

$$\begin{aligned} u(x, y, t) &= \Big(C_0 \cos(\sqrt{\lambda_n + \mu_m}\, ct) + D_0 \sin(\sqrt{\lambda_n + \mu_m}\, ct)\Big)\, \phi_n(x)\psi_m(y) \\ &= \Big(C_0 \cos(\sqrt{(n\pi/L)^2 + (m\pi/H)^2}\, ct) + D_0 \sin(\sqrt{(n\pi/L)^2 + (m\pi/H)^2}\, ct)\Big) \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}. \end{aligned}$$

A superposition of these solutions is a double series

$$\begin{aligned} u(x, y, t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Big(&C_{nm} \cos(\sqrt{(n\pi/L)^2 + (m\pi/H)^2}\, ct) \\ &+ D_{nm} \sin(\sqrt{(n\pi/L)^2 + (m\pi/H)^2}\, ct)\Big) \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}, \end{aligned}$$

where C_{nm}, D_{nm} are constants. Substituting the series into the initial conditions $u(x, y, 0) = 0$ and $\frac{\partial u}{\partial t}(x, y, 0) = f(x, y)$, we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{nm} \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H} = 0,$$

$$f(x, y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} D_{nm} \sqrt{(n\pi/L)^2 + (m\pi/H)^2}\, c \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}.$$

It follows that $C_{nm} = 0$ while $D_{nm} = \dfrac{b_{nm}}{\sqrt{(n\pi/L)^2 + (m\pi/H)^2}\, c}$, where b_{nm} are coefficients of the expansion

$$f(x, y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{nm} \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}.$$