Homework assignment #6
(due Wednesday, October 25)

Problem 1. Solve the following boundary value problem for Laplace's equation in an ellipse

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y) \in D,$$
$$u(x, y) = f(x), \quad (x, y) \in \partial D,$$

where

$$D = \left\{ (x, y) : \left(\frac{x}{b_1}\right)^2 + \left(\frac{y}{b_2}\right)^2 \le 1 \right\}, \quad b_1 > b_2 > 0$$

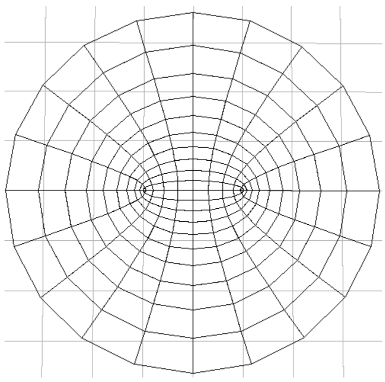
and f is a continuous function on $[-b_1, b_1]$.

To solve the problem, you will have to use elliptic coordinates (μ, ν) defined by

$$x = a \cosh \mu \cos \nu, \quad y = a \sinh \mu \sin \nu,$$

where $a > 0$ is a constant. Here $0 \le \mu < \infty$, $-\pi < \nu \le \pi$.

Coordinate lines of the elliptic coordinates are confocal ellipses and hyperbolas with focuses at points $(a, 0)$ and $(-a, 0)$. For a suitably chosen $a > 0$, the ellipse $(x/b_1)^2 + (y/b_2)^2 = 1$ is the coordinate line $\mu = M$.



(i) Find relations between a, M and b_1, b_2. Solve for a and M if possible.
(ii) Which points in the plane are singular for the elliptic coordinates?
(iii) Find the formula for Laplace's operator in elliptic coordinates.

In elliptic coordinates, the boundary condition is given by $u(M, \nu) = g(\nu)$, where g is a continuous function on $(-\pi, \pi]$.

(iv) Find the function g assuming $f(x)$ is given.
(v) Using the fact that the solution $u(x, y)$ of the boundary value problem is unique, show that $u(x, y)$ is even as a function of y.

Hint: Show that $u(x, -y)$ is also a solution.

(vi) Show that $u(x, y)$ satisfies the boundary condition

$$\frac{\partial u}{\partial y}(x, 0) = 0 \quad (-b_1 < x < b_1).$$

(vii) Show that the solution $u(\mu, \nu)$ is even as a function of ν.
(viii) Show that $u(\mu, \nu)$ satisfies the boundary conditions

$$\frac{\partial u}{\partial \nu}(\mu, 0) = \frac{\partial u}{\partial \nu}(\mu, \pi) = 0 \quad (0 < \mu < M),$$

and

$$\frac{\partial u}{\partial \mu}(0, \nu) = 0 \quad (0 < \nu < \pi).$$

(ix) Solve the following boundary value problem for Laplace's equation in a half-ellipse

$$\frac{\partial^2 u}{\partial \mu^2} + \frac{\partial^2 u}{\partial \nu^2} = 0 \quad (0 < \mu < M,\ 0 < \nu < \pi),$$

$$\frac{\partial u}{\partial \nu}(\mu, 0) = \frac{\partial u}{\partial \nu}(\mu, \pi) = 0 \quad (0 < \mu < M),$$

$$\frac{\partial u}{\partial \mu}(0, \nu) = 0, \quad u(M, \nu) = g(\nu) \quad (0 < \nu < \pi).$$

(x) Find a formula for the solution of Problem 1 in elliptic coordinates and, if possible, in Cartesian coordinates.