Math 412-501

Theory of Partial Differential Equations

# Lecture 10: Fourier series (continued). Gibbs' phenomenon.

# Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

To each integrable function $f : [-L, L] \to \mathbb{R}$ we associate a Fourier series such that

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

and for $n \geq 1$,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

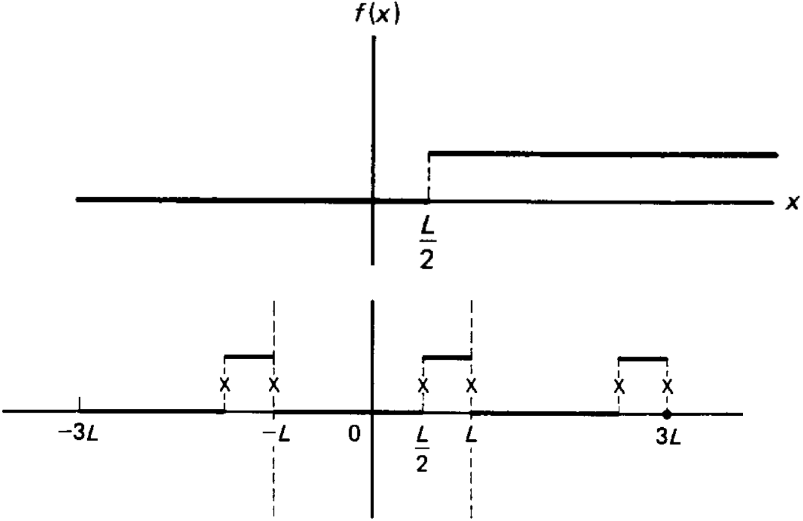
# Convergence theorem

Suppose $f : [-L, L] \to \mathbb{R}$ is a **piecewise smooth** function.

Let $F : \mathbb{R} \to \mathbb{R}$ be the $2L$-**periodic extension** of $f$.

**Theorem** The Fourier series of the function $f$ converges everywhere. The sum at a point $x$ is equal to $F(x)$ if $F$ is continuous at $x$. Otherwise the sum is equal to

$$\frac{F(x-) + F(x+)}{2}.$$




Function and its Fourier series

# Fourier sine and cosine series

Suppose $f(x)$ is an integrable function on $[0, L]$.
The Fourier sine series of $f$

$$\sum_{n=1}^{\infty} B_n \sin \tfrac{n\pi x}{L}$$

and the Fourier cosine series of $f$

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \tfrac{n\pi x}{L}$$

are defined as follows:

$$B_n = \tfrac{2}{L} \int_0^L f(x) \sin \tfrac{n\pi x}{L}\, dx;$$

$$A_0 = \tfrac{1}{L} \int_0^L f(x)\, dx, \quad A_n = \tfrac{2}{L} \int_0^L f(x) \cos \tfrac{n\pi x}{L}\, dx, \quad n \geq 1.$$

**Proposition** (i) The Fourier series of an odd function $f : [-L, L] \to \mathbb{R}$ coincides with its Fourier sine series on $[0, L]$.

(ii) The Fourier series of an even function $f : [-L, L] \to \mathbb{R}$ coincides with its Fourier cosine series on $[0, L]$.

Conversely, the Fourier sine series of a function $f : [0, L] \to \mathbb{R}$ is the Fourier series of its **odd extension** to $[-L, L]$.

The Fourier cosine series of $f$ is the Fourier series of its **even extension** to $[-L, L]$.

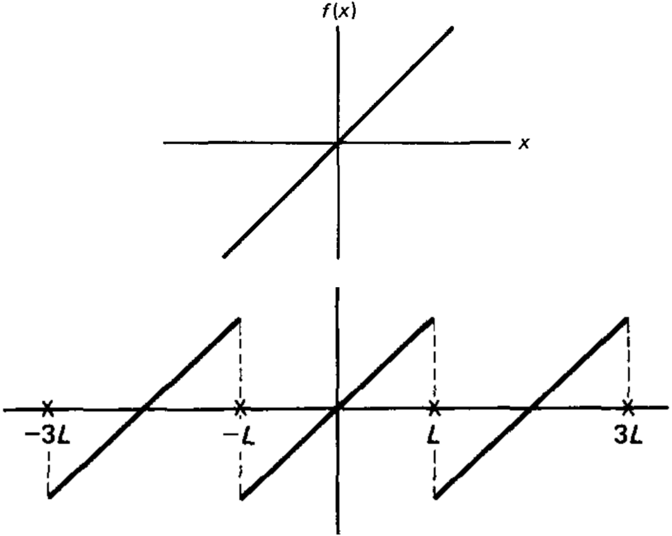
## Example

$f(x) = x$

- Fourier series $(-L \le x \le L)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = 0, \quad a_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{L}\,dx = 0.$$

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} x\sin\frac{n\pi x}{L}\,dx = \frac{L}{\pi^2}\int_{-L}^{L} \frac{\pi x}{L}\sin\frac{n\pi x}{L}\,d\left(\frac{\pi x}{L}\right) \\ &= \frac{L}{\pi^2}\int_{-\pi}^{\pi} y\sin ny\,dy = -\frac{L}{n\pi^2}\int_{-\pi}^{\pi} y\,d(\cos ny) \\ &= -\frac{L}{n\pi^2}\,y\cos ny\,\Big|_{-\pi}^{\pi} + \frac{L}{n\pi^2}\int_{-\pi}^{\pi}\cos ny\,dy \\ &= -\frac{L}{n\pi^2}\cdot 2\pi\cos n\pi = (-1)^{n+1}\frac{2L}{n\pi}. \end{aligned}$$




Fourier series of $f(x) = x$

For any $-L < x < L$,

$$\boxed{x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}}$$

For $x = L/2$ we obtain:

$$\frac{L}{2} = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2}.$$

$$\Longrightarrow \qquad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$f(x) = x$

- Fourier sine series $(0 \leq x \leq L)$ is the same as the Fourier series on $-L \leq x \leq L$.

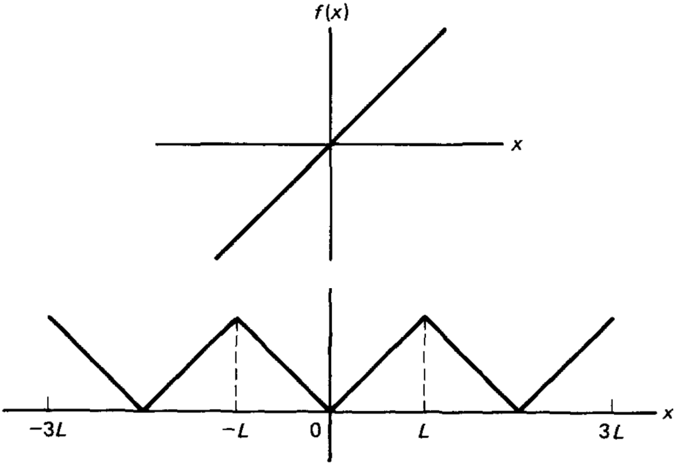
- Fourier cosine series $(0 \leq x \leq L)$

$$A_0 = \frac{1}{L} \int_0^L x \, dx = \frac{1}{L} \cdot \frac{L^2}{2} = \frac{L}{2}.$$

For $n \geq 1$,

$$A_n = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} \, dx = \frac{2L}{(n\pi)^2} (\cos n\pi - 1).$$

$A_n = 0$ if $n > 0$ is even; $A_n = -\frac{4L}{(n\pi)^2}$ if $n$ is odd.




Fourier cosine series of $f(x) = x$

For any $0 \leq x \leq L$,

$$\boxed{x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \cos \frac{(2m-1)\pi x}{L}}$$

For $x = L$ we obtain:

$$L = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \cos(2m-1)\pi.$$

$$\Longrightarrow \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

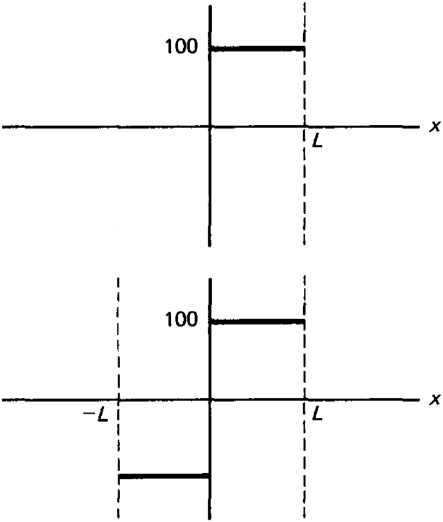
## Another example

$f(x) = 100$

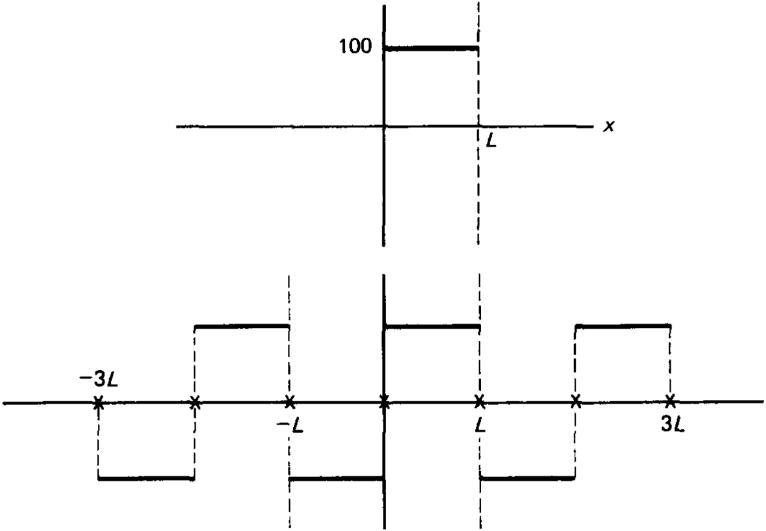
- Fourier series $(-L \leq x \leq L)$ coincides with $f(x)$.
- Fourier cosine series $(0 \leq x \leq L)$ also coincides with $f(x)$.
- Fourier sine series $(0 \leq x \leq L)$

$$B_n = \frac{2}{L} \int_0^L 100 \sin \frac{n\pi x}{L} \, dx = \frac{200}{n\pi}(1 - \cos n\pi).$$

$B_n = 0$ if $n$ is even; $B_n = \frac{400}{n\pi}$ if $n$ is odd.




Odd extension




Fourier sine series of $f(x) = 100$

For any $0 < x < L$,

$$\boxed{100 = \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin \frac{(2m-1)\pi x}{L}}$$

Partial sums:

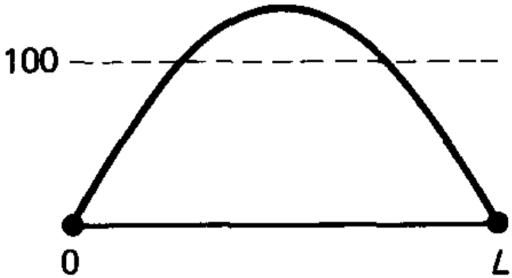
$$p_1(x) = \frac{400}{\pi} \sin \frac{\pi x}{L},$$

$$p_2(x) = \frac{400}{\pi} \left( \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} \right),$$

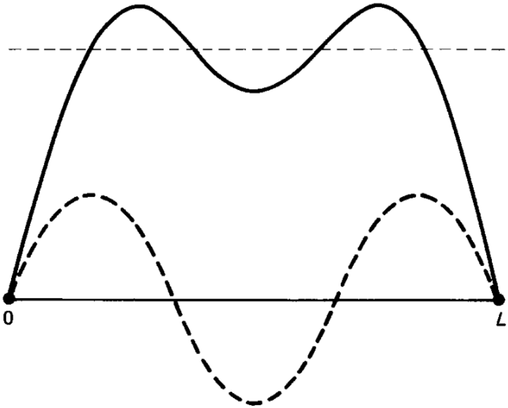
$$p_3(x) = \frac{400}{\pi} \left( \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} \right), \ \ldots$$

$$\lim_{n \to \infty} p_n(x) = 100 \ \text{ for } 0 < x < L,\ 2L < x < 3L, \ldots$$

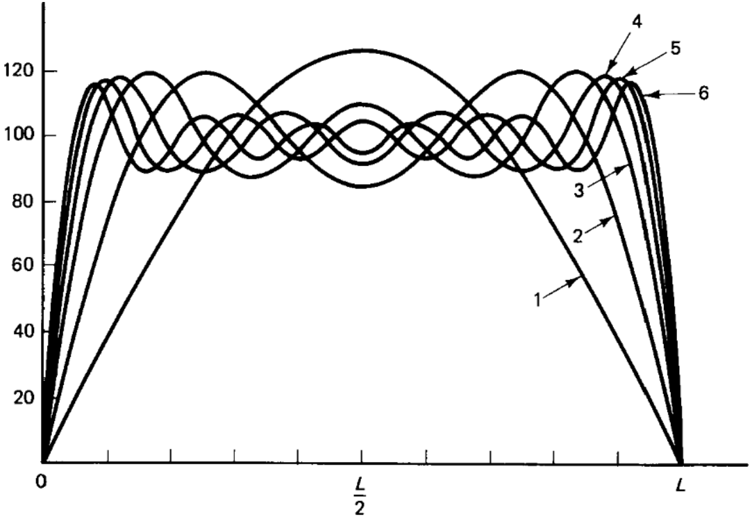
$$\lim_{n \to \infty} p_n(x) = -100 \ \text{ for } -L < x < 0,\ L < x < 2L, \ldots$$




$p_1(x)$




$p_2(x)$




$p_n(x), \ 1 \leq n \leq 6.$

## Gibbs' phenomenon

The partial sum $p_n(x)$ attains its maximal value $v_n$ on the interval $0 \le x \le L$ at two points $x_n^+$, $x_n^-$ such that $x_n^+ \to L$ and $x_n^- \to 0$ as $n \to \infty$.

Actually, $x_n^- = \frac{L}{2n}$, $x_n^+ = L - \frac{L}{2n}$.

The maximal **overshoot** $v_n = p_n(x_n^\pm)$ satisfies $v_1 > v_2 > v_3 > \dots$ and $\lim\limits_{n\to\infty} v_n = v_\infty > \mathbf{100}$.

$$\text{Actually, } v_\infty = \frac{200}{\pi} \int_0^\pi \frac{\sin y}{y}\, dy \approx 117.898$$

The **Gibbs phenomenon** occurs for any piecewise smooth function at any discontinuity. The ultimate overshoot rate of $\approx 9\%$ of the jump is universal.

# Term-by-term differentiation

Fourier cosine series of $f_1(x) = x$:

$$\frac{L}{2} - \frac{4L}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \cos \frac{(2m-1)\pi x}{L}$$

Fourier sine series of $f_2(x) = 1$:

$$\frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin \frac{(2m-1)\pi x}{L}$$

The second series can be obtained by term-by-term differentiation of the first series.
And, by the way, $f_1'(x) = f_2(x)$.

**Theorem** Suppose that a function $f : [-L, L] \to \mathbb{R}$ is continuous, piecewise smooth, and $f(-L) = f(L)$.

Then the Fourier series of $f'$ (on $[-L, L]$) can be obtained via term-by-term differentiation of the Fourier series of $f$.

Let $f : [0, L] \to \mathbb{R}$ be a continuous function and $F : [-L, L] \to \mathbb{R}$ be its even extension. Then $F$ is also continuous and $F(-L) = F(L)$. If $f$ is piecewise smooth, so is $F$. Moreover, $F'$ is the odd extension of $f'$ to $[-L, L]$.

**Corollary** Let $f : [0, L] \to \mathbb{R}$ be a continuous, piecewise smooth function. Then the term-by-term differentiation of the Fourier cosine series of $f$ yields the Fourier sine series of $f'$.

*Example.* Find the Fourier series of $f(x) = x^2$.

$$x^2 \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Term-by-term differentiation yields

$$-\sum_{n=1}^{\infty} a_n \frac{n\pi}{L} \sin \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos \frac{n\pi x}{L}.$$

By the theorem, this should be the Fourier series of $f'(x) = 2x$, which is

$$2x \sim \frac{4L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}.$$

Hence $b_n = 0$ and $a_n = (-1)^n \frac{4L^2}{n^2\pi^2}$ for $n \geq 1$.

It remains to find $a_0 = \frac{1}{2L} \int_{-L}^{L} x^2 \, dx = \frac{L^2}{3}$.

# Term-by-term integration

**Theorem** Suppose that a piecewise continuous function $f : [-L, L] \to \mathbb{R}$ has the Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

Then

$$\int_c^x f(y)\,dy = \int_c^x a_0\,dy$$

$$+ \sum_{n=1}^{\infty} \int_c^x a_n \cos \frac{n\pi y}{L}\,dy + \sum_{n=1}^{\infty} \int_c^x b_n \sin \frac{n\pi y}{L}\,dy.$$

for any interval $[c, x] \subset [-L, L]$.

*Term-by-term integration is always possible but the result need not be a Fourier series.*