Math 412-501

Theory of Partial Differential Equations

# Lecture 5: Linearity and homogeneity.

## Linearity

Linear space = a set $V$ of objects that can be summed and multiplied by scalars (real numbers).

That is, for any $u, v \in V$ and $r \in \mathbb{R}$ expressions

$$\boxed{u + v} \text{ and } \boxed{ru}$$

should make sense.

Certain restrictions apply. For instance,

$$u + v = v + u,$$
$$u + u = 2u.$$

Given $u_1, u_2, \dots, u_k \in V$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$,

$$\boxed{r_1 u_1 + r_2 u_2 + \cdots + r_k u_k}$$

is called a **linear combination** of $u_1, u_2, \dots, u_k$.

# Examples

- $\mathbb{R}$: real numbers
- $\mathbb{Z}$: integers (**not** a linear space)
- $\mathbb{R}^n$ $(n > 1)$: $n$-dimensional vectors
- $\mathbb{C}$: complex numbers

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$
- $F(\mathbb{R}) \setminus C(\mathbb{R})$: all discontinuous functions

$f : \mathbb{R} \to \mathbb{R}$ (**not** a linear space)
- $C^1[a, b]$: all continuously differentiable functions

$f : [a, b] \to \mathbb{R}$
- $C^\infty[a, b]$: all smooth functions $f : [a, b] \to \mathbb{R}$

# More examples

- $C^2([a, b] \times [c, d])$: twice continuously differentiable functions $u = u(x, t)$, $a \leq x \leq b$, $c \leq t \leq d$
- $\{u \in C^2([a, b] \times [c, d]) : u(a, t) = u(b, t) = 0\}$: twice continuously differentiable functions satisfying Dirichlet boundary conditions
- $L[a, b]$: integrable functions $f : [a, b] \to \mathbb{R}$; $\int_a^b |f(x)|\, dx < \infty$
- $L^2[a, b]$: square-integrable functions $f : [a, b] \to \mathbb{R}$; $\int_a^b |f(x)|^2\, dx < \infty$

Note that $|f(x) + g(x)|^2 \leq 2|f(x)|^2 + 2|g(x)|^2$.

## Linear maps

Given linear spaces $V_1$ and $V_2$, a map $A : V_1 \to V_2$ is **linear** if

$$A(v + u) = A(v) + A(u),$$
$$A(ru) = rA(u)$$

for any $u, v \in V_1$ and $r \in \mathbb{R}$.

A linear map $\ell : V \to \mathbb{R}$ is called a **linear functional** on $V$.

If $V_1 = V_2$ (or if both $V_1$ and $V_2$ are functional spaces) then a linear map $L : V_1 \to V_2$ is called a **linear operator**.

# Linear functionals

- $V = \mathbb{R}^n, \;\; \ell(v) = (v, v_0)$, where $v_0 \in V$.
- $V = C[a, b], \;\; \ell(f) = f(a)$.
- $V = C^1[a, b], \;\; \ell(f) = f'(b)$.
- $V = C[a, b], \;\; \ell(f) = \int_a^b f(x)\, dx$.
- $V = C[a, b], \;\; \ell(f) = \int_a^b g(x) f(x)\, dx$,

where $g \in C[a, b]$.

## Linear operators

- $V = \mathbb{R}^n$, $L(v) = Av$, where $A$ is an $n \times n$ matrix.
- $V = C[a,b]$, $L(f) = gf$, where $g \in C[a,b]$.
- $V_1 = C^1[a,b]$, $V_2 = C[a,b]$, $L(f) = f'$.
- $V = C[a,b]$, $(L(f))(x) = \int_a^x f(\xi)\, d\xi$.
- $V = C[a,b]$, $(L(f))(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi$, where $G \in C([a,b] \times [a,b])$.
- $V_1 = C([a,b],[c,d])$, $V_2 = C[c,d]$, $(L(u))(t) = u(a,t)$.

## Linear differential operators

- ordinary differential operator:

$$L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2 \quad (g_0, g_1, g_2 \text{ are functions})$$

- heat operator: $L = \dfrac{\partial}{\partial t} - k\,\dfrac{\partial^2}{\partial x^2}$

- wave operator: $L = \dfrac{\partial^2}{\partial t^2} - c^2\,\dfrac{\partial^2}{\partial x^2}$

(a.k.a. the d'Alembertian; denoted by $\Box$).

- Laplace's operator: $L = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

(a.k.a. the Laplacian; denoted by $\Delta$ or $\nabla^2$).

# Linear equations

An equation is called **linear** if it can be written in the form

$$L(u) = f,$$

where $L : V_1 \longrightarrow V_2$ is a linear map, $f \in V_2$ is given, and $u \in V_1$ is the unknown.

An equation is called **linear homogeneous** if it can be written in the form

$$L(u) = 0,$$

where $L : V_1 \longrightarrow V_2$ is a linear map and $u \in V_1$ is the unknown.

*Remark.* $(x+1)^2 = x^2 \implies 2x = -1$ (linear)

Heat equation, wave equation, and Laplace's equation are linear homogeneous equations.

Korteweg-de Vries **(KdV)** equation:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad \text{(non-linear)}$$

Initial condition: $u(x,0) = f(x)$ (linear equation).

Boundary conditions $u(0,t) = u_0(t)$ and $\frac{\partial u}{\partial x}(0,t) = \phi(t)$ are linear equations.

Boundary conditions $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(0,t) = 0$ are linear homogeneous equations.

## Properties of linear spaces/maps/equations

**Theorem** **(i)** Suppose $V_1$ and $V_2$ are linear spaces. Then the set of all linear maps $L : V_1 \to V_2$ is also a linear space.

**(ii)** Composition of linear maps is also a linear map.

**(iii)** The set of solutions of a linear homogeneous equation is a linear space.

*How do we solve a linear homogeneous PDE?*
*Step 1:* Find some solutions.
*Step 2:* Form linear combinations of solutions obtained on Step 1.
*Step 3:* Show that every solution can be approximated by solutions obtained on Step 2.