Math 412-501

Theory of Partial Differential Equations

Lecture 7: Eigenvalue problems. Solution of the initial-boundary value problem for the heat equation.

Boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$
$$u(0,t) = u(L,t) = 0.$$

We are looking for solutions $u(x,t) = \phi(x)G(t)$.

PDE holds if

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$
$$\frac{dG}{dt} = -\lambda k G$$

for the same constant λ.

Boundary conditions hold if

$$\phi(0) = \phi(L) = 0.$$

Boundary value problem:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad 0 \le x \le L,$$
$$\phi(0) = \phi(L) = 0.$$

There is an obvious solution: 0.
When is it **not unique?**

If for some value of λ the boundary value problem has a nonzero solution ϕ, then this λ is called an **eigenvalue** and ϕ is called an **eigenfunction**.

The **eigenvalue problem** is to find all eigenvalues (and corresponding eigenfunctions).

Matrices vs. differential operators

Suppose A is an $n \times n$ matrix, $v \in \mathbb{R}^n$ is a nonzero vector, and $\lambda = \text{const}$. If $\boxed{Av = \lambda v}$ then λ is an **eigenvalue** of A and v is an **eigenvector**.

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad 0 \le x \le L,$$
$$\phi(0) = \phi(L) = 0.$$

Instead of A, we have the linear operator $-\frac{d^2}{dx^2}$.
Instead of $\mathbb{R}^n$, we have the linear space
$V = \{\phi \in C^2[0, L] : \phi(0) = \phi(L) = 0\}$.

Eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = \phi(L) = 0.$$

We are looking only for real eigenvalues.

Three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

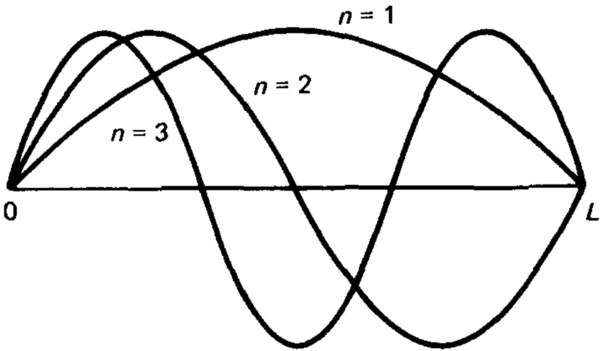
Case 1: $\lambda > 0$. $\quad \phi(x) = C_1 \cos \mu x + C_2 \sin \mu x$,
where $\lambda = \mu^2$, $\mu > 0$.

$\phi(0) = \phi(L) = 0 \implies C_1 = 0, \; C_2 \sin \mu L = 0.$

A nonzero solution exists if $\mu L = n\pi$, $n \in \mathbb{Z}$.

So $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \dots$ are eigenvalues and $\phi_n(x) = \sin \frac{n\pi x}{L}$ are corresponding eigenfunctions.

Eigenfunctions



$$\phi_n(x) = \sin \frac{n\pi x}{L}$$

Are there other eigenfunctions?

Case 2: $\lambda = 0$. $\quad \phi(x) = C_1 + C_2 x$.

$$\phi(0) = \phi(L) = 0 \implies C_1 = C_1 + C_2 L = 0$$
$$\implies C_1 = C_2 = 0.$$

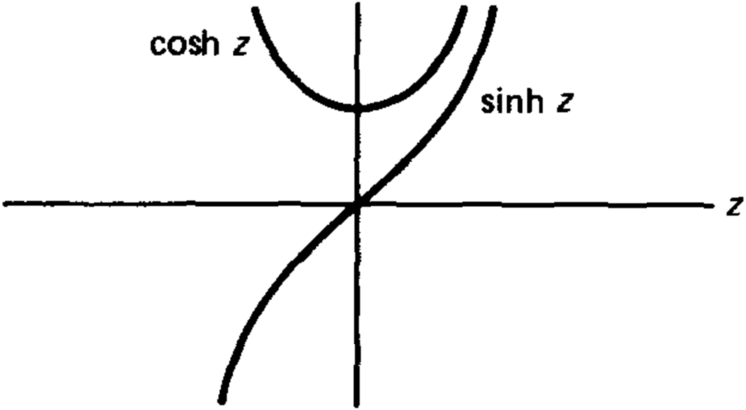
Case 3: $\lambda < 0$. $\quad \phi(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$,
where $\lambda = -\mu^2$, $\mu > 0$.

$$\boxed{\cosh z = \frac{e^z + e^{-z}}{2}} \qquad \boxed{\sinh z = \frac{e^z - e^{-z}}{2}}$$

$$e^z = \cosh z + \sinh z, \qquad e^{-z} = \cosh z - \sinh z.$$

$$\phi(x) = D_1 \cosh \mu x + D_2 \sinh \mu x, \qquad D_1, D_2 = \text{const.}$$
$$\phi(0) = 0 \implies D_1 = 0$$
$$\phi(L) = 0 \implies D_2 \sinh \mu L = 0 \implies D_2 = 0$$

Hyperbolic functions



There is another way to show that all eigenvalues are positive. Given an eigenfunction ϕ, let

$$I = \int_0^L \phi''(x)\phi(x)\,dx.$$

Since $\phi'' = -\lambda\phi$, we have

$$I = -\lambda \int_0^L |\phi(x)|^2\,dx.$$

Integrating by parts, we obtain

$$I = \phi'(L)\phi(L) - \phi'(0)\phi(0) - \int_0^L \phi'(x)\phi'(x)\,dx.$$

Hence

$$\boxed{\lambda \int_0^L |\phi(x)|^2\,dx = \int_0^L |\phi'(x)|^2\,dx.}$$

$\Longrightarrow$ **either** $\lambda > 0$, **or else** $\lambda = 0$ and $\phi = \text{const}$.

Summary

Eigenvalue problem: $\phi'' = -\lambda\phi, \;\; \phi(0) = \phi(L) = 0.$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2, \; n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(x) = \sin\frac{n\pi x}{L}.$

Solution of the heat equation: $u(x,t) = \phi(x)G(t).$

$$\frac{dG}{dt} = -\lambda kG \implies G(t) = C_0\exp(-\lambda kt)$$

Theorem For $n = 1, 2, \ldots,$ the function

$$\boxed{u(x,t) = e^{-\lambda_n kt}\phi_n(x) = \exp(-\frac{n^2\pi^2}{L^2}kt)\sin\frac{n\pi x}{L}}$$

is a solution of the following boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0.$$

Initial-boundary value problem

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$

$$u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0.$$

Function $u(x,t) = e^{-\lambda_n kt}\phi_n(x)$ is a solution of the boundary value problem. Initial condition is satisfied if $f = \phi_n$. For any $B_1, B_2, \ldots, B_N \in \mathbb{R}$ the function

$$u(x,t) = \sum_{n=1}^{N} B_n e^{-\lambda_n kt}\phi_n(x)$$

is also a solution of the boundary value problem. This time the initial condition is satisfied if

$$f(x) = \sum_{n=1}^{N} B_n\phi_n(x) = \sum_{n=1}^{N} B_n \sin\frac{n\pi x}{L}.$$

From finite sums to series

Conjecture For suitably chosen coefficients $B_1, B_2, B_3, \dots$ the function

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n kt} \phi_n(x)$$

is a solution of the boundary value problem. This solution satisfies the initial condition with

$$f(x) = \sum_{n=1}^{\infty} B_n \phi_n(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.$$

Theorem If $\sum_{n=1}^{\infty} |B_n| < \infty$ then the conjecture is true. Namely, $u(x,t)$ is smooth for $t > 0$ and solves the boundary value problem. Also, $u(x,t)$ is continuous for $t \geq 0$ and satisfies the initial condition.

How do we solve the initial-boundary value problem?

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$
$$u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0.$$

- Expand the function f into a series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.$$

- Write the solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{n^2\pi^2}{L^2}kt\right) \sin \frac{n\pi x}{L}.$$

J. Fourier, The Analytical Theory of Heat
(written in 1807, published in 1822)