Math 412-501

Theory of Partial Differential Equations

Lecture 9: Fourier series.

Trigonometric polynomial

$$p(x) = a_0 + \sum_{n=1}^{N} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{N} b_n \sin \frac{n\pi x}{L}.$$

- $p(x)$ is an infinitely smooth function
- $p(x)$ is $2L$-periodic:

$p(x) = p(x + 2L) = p(x - 2L)$ for all x.

$p(x)$ is a finite linear combination of the functions

$$1,\ \cos \frac{\pi x}{L},\ \sin \frac{\pi x}{L},\ \cos \frac{2\pi x}{L},\ \sin \frac{2\pi x}{L}, \dots$$

For any positive integers n and m:

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} \frac{1}{2}L & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} \frac{1}{2}L & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Hence

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} L & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} L & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Also,

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0.$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx = \int_{-L}^{L} \sin\frac{n\pi x}{L}\,dx = 0.$$

$$\int_{-L}^{L} 1\,dx = 2L.$$

$$p(x) = a_0 + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{L}.$$

$$\int_{-L}^{L} p(x)\,dx = a_0 \cdot 2L.$$

For $1 \leq n \leq N$,

$$\int_{-L}^{L} p(x)\cos\frac{n\pi x}{L}\,dx = a_n \cdot L.$$

$$\int_{-L}^{L} p(x)\sin\frac{n\pi x}{L}\,dx = b_n \cdot L.$$

$$p(x) = a_0 + \sum_{n=1}^{N} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{N} b_n \sin \frac{n\pi x}{L}.$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} p(x)\, dx.$$

For $1 \leq n \leq N$,

$$a_n = \frac{1}{L} \int_{-L}^{L} p(x) \cos \frac{n\pi x}{L}\, dx.$$

$$b_n = \frac{1}{L} \int_{-L}^{L} p(x) \sin \frac{n\pi x}{L}\, dx.$$

Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Suppose $f : [-L, L] \to \mathbb{R}$ is an integrable function. Let

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

and for $n \geq 1$,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

Then we obtain the Fourier series of f (associated to f) on the interval $[-L, L]$.

Questions

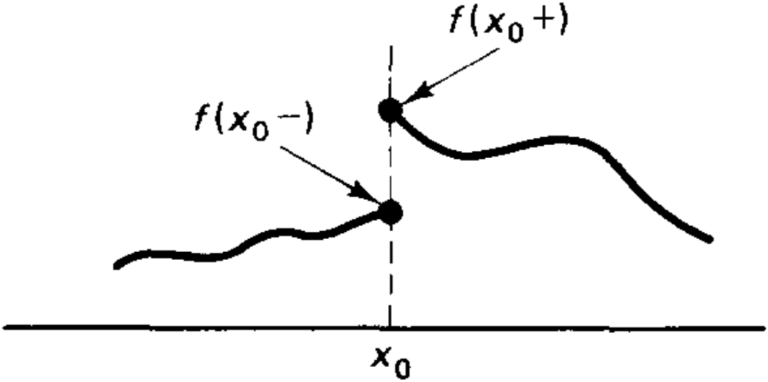
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

- When does a Fourier series converge everywhere? When does it converge uniformly?
- If a Fourier series does not converge everywhere, then what is the set of points where it converges?
- If a Fourier series is associated to a function, then how do convergence properties depend on the function?
- If a Fourier series is associated to a function, then how does the sum of the series relate to the function?

Answers

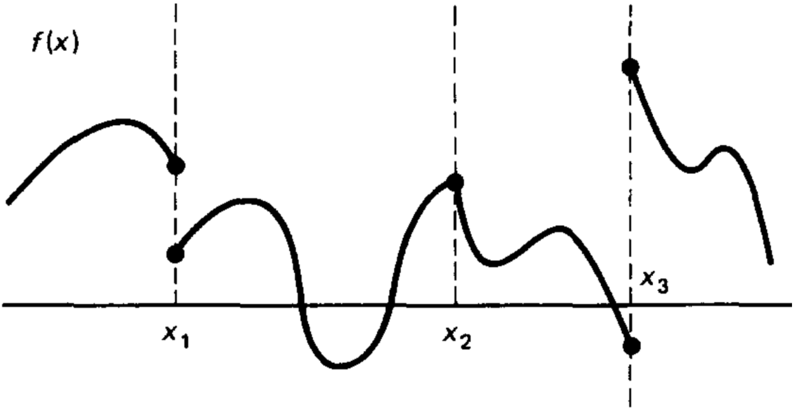
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

- Complete answers are never easy (and hardly possible) when dealing with the Fourier series!
- A Fourier series converges everywhere provided that $a_n \to 0$ and $b_n \to 0$ fast enough (however fast decay is not necessary).
- The Fourier series of a continuous function converges to this function **almost everywhere**.
- The Fourier series associated to a function converges everywhere provided that the function is **piecewise smooth** (condition may be relaxed).

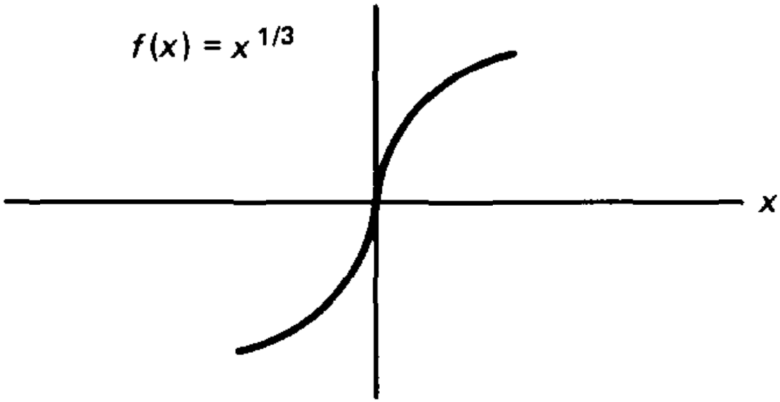


Jump discontinuity

Piecewise continuous = finitely many jump discontinuities



Piecewise smooth function
(both function and its derivative
are piecewise continuous)



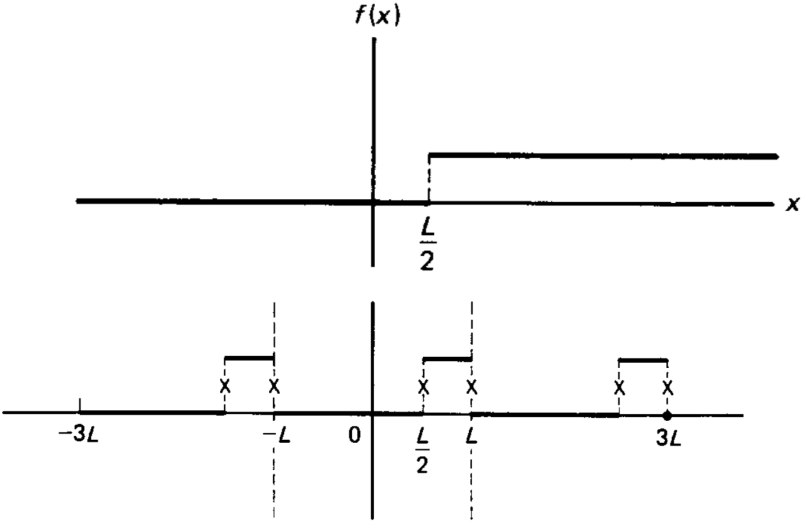
Continuous, but not piecewise smooth function

Convergence theorem

Suppose $f : [-L, L] \to \mathbb{R}$ is a piecewise smooth function. Let $F : \mathbb{R} \to \mathbb{R}$ be the $2L$-**periodic extension** of f. That is, F is $2L$-periodic and $F(x) = f(x)$ for $-L < x \leq L$. Clearly, F is also piecewise smooth.

Theorem The Fourier series of the function f converges everywhere. The sum at a point x is equal to $F(x)$ if F is continuous at x. Otherwise the sum is equal to

$$\frac{F(x-) + F(x+)}{2}.$$



Function and its Fourier series

Fourier sine and cosine series

Suppose $f(x)$ is an integrable function on $[0, L]$.
The Fourier sine series of f

$$\sum_{n=1}^{\infty} B_n \sin \tfrac{n\pi x}{L}$$

and the Fourier cosine series of f

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \tfrac{n\pi x}{L}$$

are defined as follows:

$$B_n = \tfrac{2}{L} \int_0^L f(x) \sin \tfrac{n\pi x}{L} \, dx;$$

$$A_0 = \tfrac{1}{L} \int_0^L f(x) \, dx, \qquad A_n = \tfrac{2}{L} \int_0^L f(x) \cos \tfrac{n\pi x}{L} \, dx, \quad n \geq 1.$$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \quad n \geq 1,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

If f is **odd**, $f(-x) = -f(x)$, then $a_n = 0$ and

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

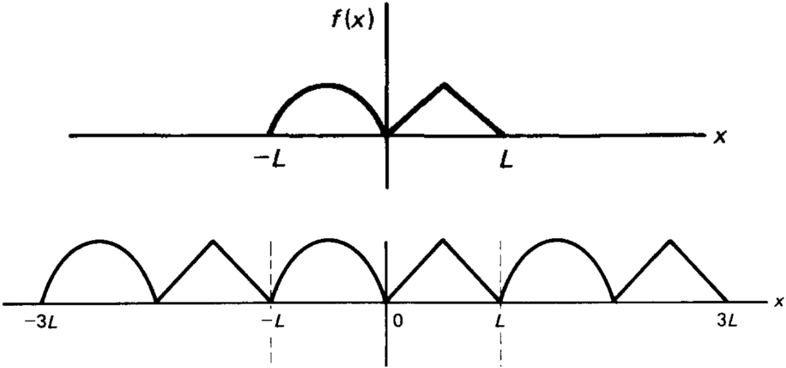
Similarly, if f is **even**, $f(-x) = f(x)$, then $b_n = 0$ and $a_n = A_n$.

Proposition (i) The Fourier series of an odd function $f : [-L, L] \to \mathbb{R}$ coincides with its Fourier sine series on $[0, L]$.

(ii) The Fourier series of an even function $f : [-L, L] \to \mathbb{R}$ coincides with its Fourier cosine series on $[0, L]$.

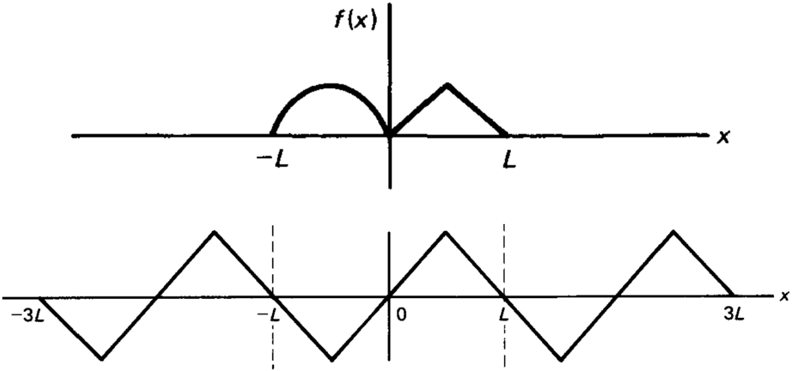
Conversely, the Fourier sine series of a function $f : [0, L] \to \mathbb{R}$ is the Fourier series of its **odd extension** to $[-L, L]$.

The Fourier cosine series of f is the Fourier series of its **even extension** to $[-L, L]$.

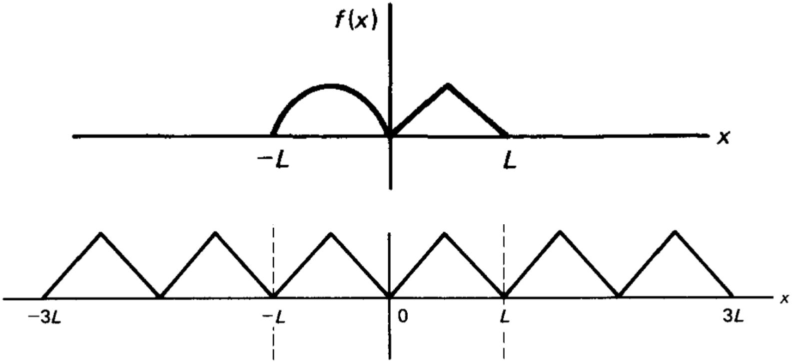


Fourier series

(2L-periodic)



Fourier sine series
(2L-periodic and odd)



Fourier cosine series
(2L-periodic and even)