Math 412-501

Theory of Partial Differential Equations

Lecture 2-1:
Higher-dimensional heat equation.

PDEs: two variables

heat equation: $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

PDEs: three variables

heat equation: $$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

One-dimensional heat equation

Describes heat conduction in a rod:

$$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q}$$

$K_0 = K_0(x)$, $c = c(x)$, $\rho = \rho(x)$, $Q = Q(x, t)$.

Assuming K_0, c, ρ are constant (uniform rod) and $Q = 0$ (no heat sources), we obtain

$$\boxed{\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}}$$

where $k = K_0(c\rho)^{-1}$.

Heat conduction in three dimensions

$u(x, y, z, t) =$
temperature at point (x, y, z) at time t

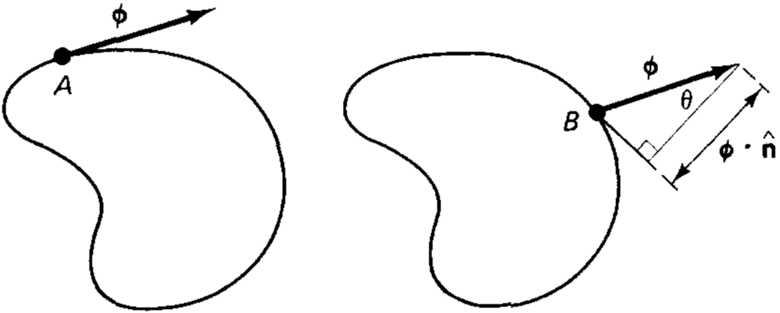
$e(x, y, z, t) =$ thermal energy density (thermal energy per unit volume)

$Q(x, y, z, t) =$ density of heat sources (heat energy per unit volume generated per unit time)

$\phi(x, y, z, t) =$ heat flux
$\vec{\phi}(x, y, z, t)$ is a vector field

thermal energy flowing per unit surface per unit time $= \vec{\phi}(x, y, z, t) \cdot \vec{\mathbf{n}}(x, y, z)$, where $\mathbf{n}(x, y, z)$ is the unit normal vector of the surface

Heat flux



$c(x, y, z) =$ specific heat or heat capacity (the heat energy supplied to a unit mass of a substance to raise its temperature one unit)

$\rho(x, y, z) =$ mass density (mass per unit volume)

Thermal energy in a volume is equal to the energy it takes to raise the temperature of the volume from a reference temperature (zero) to its actual temperature.

$$e(x, y, z, t) \cdot \Delta V = c(x, y, z) u(x, y, z, t) \cdot \rho(x, y, z) \cdot \Delta V$$

$$\boxed{e(x, y, z, t) = c(x, y, z) \rho(x, y, z) u(x, y, z, t)}$$

Four quantities: u, e, Q, ϕ.
Heat equation should involve only two: u and Q.

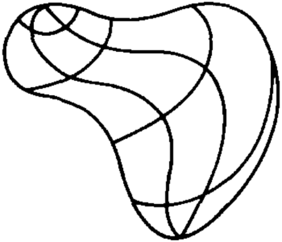
Heat equation is derived from two physical laws:

- conservation of heat energy,
- Fourier's low of heat conduction.

Conservation of heat energy (in a volume in a period of time):

change of heat energy	$=$	heat energy flowing across boundary	$+$	heat energy generated inside

rate of change of heat energy	$=$	heat energy flowing across boundary per unit time	$+$	heat energy generated inside per unit time



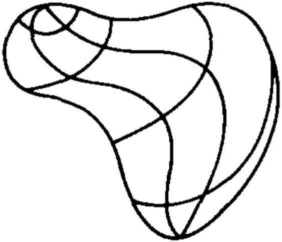
subregion R

heat energy:

$$\iiint_R e(x,y,z,t)\,dx\,dy\,dz = \iiint_R e\,dV$$

rate of change of heat energy:

$$\frac{\partial}{\partial t}\Big(\iiint_R e\,dV\Big) = \frac{\partial}{\partial t}\Big(\iiint_R c\rho u\,dV\Big)$$



subregion R

heat energy flowing across boundary per unit time:

$$-\oiint_{\partial R} \vec{\phi} \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the unit outward normal vector of ∂R.

heat energy generated inside per unit time:

$$\iiint_R Q\, dV$$

$$\frac{\partial}{\partial t}\Big(\iiint_R c\rho u\, dV\Big) = -\oiint_{\partial R} \vec{\phi}\cdot\mathbf{n}\, dS + \iiint_R Q\, dV$$

$$\iiint_R c\rho\frac{\partial u}{\partial t}\, dV = -\oiint_{\partial R} \vec{\phi}\cdot\mathbf{n}\, dS + \iiint_R Q\, dV$$

$$\boxed{\iiint_R \nabla\cdot\vec{\phi}\, dV = \oiint_{\partial R} \vec{\phi}\cdot\mathbf{n}\, dS}$$

where $\vec{\phi} = (\phi_x, \phi_y, \phi_z)$, $\nabla\cdot\vec{\phi} = \frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y} + \frac{\partial\phi_z}{\partial z}$.

(Gauss' formula) (divergence theorem)

$\nabla\cdot\vec{\phi}$ is called the **divergence** of vector field ϕ.

$$\iiint_R c\rho \frac{\partial u}{\partial t}\, dV = -\iiint_R \nabla \cdot \vec{\phi}\, dV + \iiint_R Q\, dV$$

Since R is an arbitrary subregion,

$$\boxed{c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \vec{\phi} + Q}$$

Fourier's law of heat conduction:

$$\vec{\phi} = -K_0 \nabla u,$$

where $K_0 = K_0(x, u)$ is the *thermal conductivity* and $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z})$ is the *gradient* of u.

Heat equation: $$\boxed{c\rho\frac{\partial u}{\partial t} = \nabla\cdot(K_0\,\nabla u) + Q}$$

Assuming $K_0 = \text{const}$, we have

$$c\rho\frac{\partial u}{\partial t} = K_0\,\nabla^2 u + Q,$$

where $\nabla^2 u = \nabla\cdot(\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ is the **Laplacian** of u.

Assuming $K_0, c, \rho = \text{const}$ (uniform medium) and $Q = 0$ (no heat sources), we obtain

$$\boxed{\frac{\partial u}{\partial t} = k\,\nabla^2 u,}$$

where $k = K_0(c\rho)^{-1}$ is called the *thermal diffusivity*.

Notation

Each function $f : \mathbb{R}^3 \to \mathbb{R}$ is assigned the gradient (a vector field) and the Laplacian (a function). Each vector field $\vec{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$ is assigned the divergence (a function).

"physical" notation: $\quad \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

gradient: $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$

divergence: $\nabla \cdot \vec{\phi} = \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z}$

Laplacian: $\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

"mathematical" notation:

gradient: $\operatorname{grad} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$

divergence: $\operatorname{div} \vec{\phi} = \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z}$

Laplacian: $\Delta f = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$