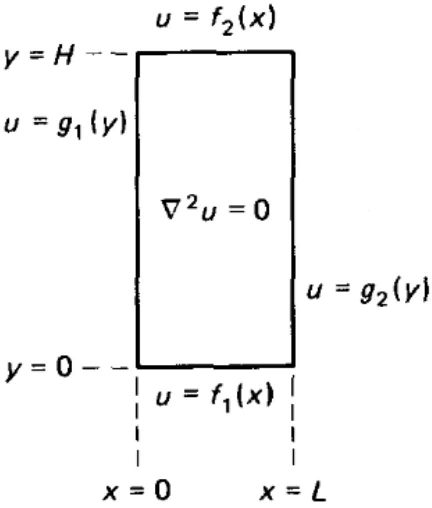
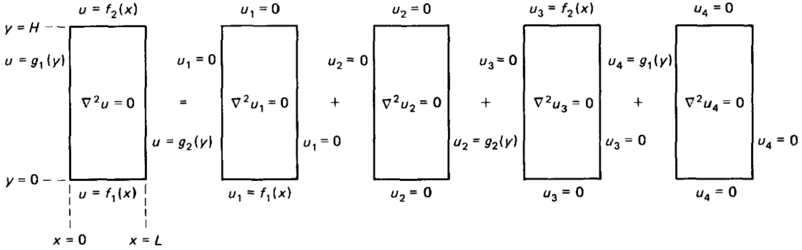
Math 412-501

Theory of Partial Differential Equations

Lecture 2-4:
Laplace's equation in polar coordinates.

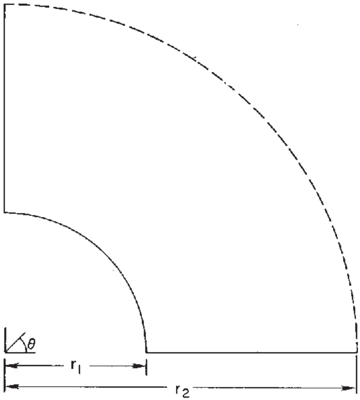
Laplace's equation in a rectangle





$u = f_2(x)$
$y = H$
$u = g_1(y)$
$\nabla^2 u = 0$
$u = g_2(y)$
$y = 0$
$u = f_1(x)$
$x = 0$
$x = L$
=
$u_1 = 0$
$u_1 = 0$
$\nabla^2 u_1 = 0$
$u_1 = 0$
$u_1 = f_1(x)$
+
$u_2 = 0$
$u_2 = 0$
$\nabla^2 u_2 = 0$
$u_2 = g_2(y)$
$u_2 = 0$
+
$u_3 = f_2(x)$
$u_3 = 0$
$\nabla^2 u_3 = 0$
$u_3 = 0$
$u_3 = 0$
+
$u_4 = 0$
$u_4 = g_1(y)$
$\nabla^2 u_4 = 0$
$u_4 = 0$
$u_4 = 0$

Chunk of an annulus



In polar coordinates: $r_1 < r < r_2, \;\; 0 < \theta < \frac{\pi}{2}$

Laplace's equation in polar coordinates

In Cartesian coordinates (x, y),

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Polar coordinates (r, θ). Transition formulas:

$$\boxed{x = r\cos\theta, \qquad y = r\sin\theta.}$$

Jacobian matrix: $\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

Jacobian determinant: $\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$

Inverse matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse Jacobian matrix:

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}^{-1} = \frac{1}{r} \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary smooth function.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta \cdot \frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta \cdot \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta \cdot \frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta \cdot \frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \left(\cos\theta \cdot \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \cdot \frac{\partial}{\partial \theta}\right)\left(\cos\theta \cdot \frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta \cdot \frac{\partial u}{\partial \theta}\right)$$

$$\begin{aligned} &= \cos^2\theta \cdot \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\cos\theta\sin\theta \cdot \frac{\partial u}{\partial \theta} - \frac{1}{r}\cos\theta\sin\theta \cdot \frac{\partial^2 u}{\partial r\,\partial \theta} \\ &\quad + \frac{1}{r}\sin^2\theta \cdot \frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\cos\theta \cdot \frac{\partial^2 u}{\partial \theta\,\partial r} \\ &\quad + \frac{1}{r^2}\sin\theta\cos\theta \cdot \frac{\partial u}{\partial \theta} + \frac{1}{r^2}\sin^2\theta \cdot \frac{\partial^2 u}{\partial \theta^2}. \end{aligned}$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\sin\theta \cdot \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \cdot \frac{\partial}{\partial \theta}\right)\left(\sin\theta \cdot \frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta \cdot \frac{\partial u}{\partial \theta}\right)$$

$$\begin{aligned} &= \sin^2\theta \cdot \frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\sin\theta\cos\theta \cdot \frac{\partial u}{\partial \theta} + \frac{1}{r}\sin\theta\cos\theta \cdot \frac{\partial^2 u}{\partial r\,\partial \theta} \\ &\quad + \frac{1}{r}\cos^2\theta \cdot \frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\sin\theta \cdot \frac{\partial^2 u}{\partial \theta\,\partial r} \\ &\quad - \frac{1}{r^2}\cos\theta\sin\theta \cdot \frac{\partial u}{\partial \theta} + \frac{1}{r^2}\cos^2\theta \cdot \frac{\partial^2 u}{\partial \theta^2}. \end{aligned}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2}$$

Laplace's equation in polar coordinates:

$$\boxed{\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0}$$

or

$$\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} = 0}$$

Separation of variables: $u(r,\theta) = h(r)\phi(\theta)$.
Substitute this into Laplace's equation:

$$\frac{d^2h}{dr^2}\,\phi(\theta) + \frac{1}{r}\,\frac{dh}{dr}\,\phi(\theta) + \frac{1}{r^2}\,h(r)\,\frac{d^2\phi}{d\theta^2} = 0.$$

Divide both sides by $r^{-2}h(r)\phi(\theta) = r^{-2}u(r,\theta)$:

$$\frac{1}{h}\cdot\left(r^2\,\frac{d^2h}{dr^2} + r\,\frac{dh}{dr}\right) = -\frac{1}{\phi}\cdot\frac{d^2\phi}{d\theta^2}.$$

It follows that

$$\frac{1}{h}\cdot\left(r^2\,\frac{d^2h}{dr^2} + r\,\frac{dh}{dr}\right) = -\frac{1}{\phi}\cdot\frac{d^2\phi}{d\theta^2} = \lambda = \text{const}.$$

The variables have been separated:

$$r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} = \lambda h,$$

$$\frac{d^2 \phi}{d\theta^2} = -\lambda \phi.$$

Proposition Suppose h and ϕ are solutions of the above ODEs for the same value of λ. Then $u(r, \theta) = h(r)\phi(\theta)$ is a solution of Laplace's equation.

Euler's (or equidimensional) equation

$$\boxed{r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} - \lambda h = 0 \quad (r > 0)}$$

Suppose $h(r) = r^p$, $p \in \mathbb{R}$. Then

$$h'(r) = pr^{p-1}, \qquad h''(r) = p(p-1)r^{p-2}.$$

Hence

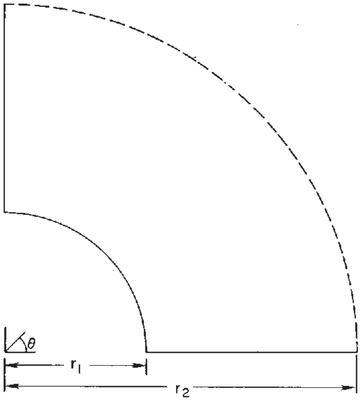
$$r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} = \big(p(p-1) + p\big) r^p = p^2 r^p.$$

$$\lambda > 0 \implies h(r) = C_1 r^p + C_2 r^{-p} \quad (\lambda = p^2,\ p > 0)$$

$$\lambda = 0 \implies r^2 h''(r) + rh'(r) = 0 \implies r(rh')' = 0$$

$$\implies rh'(r) = C_2 \implies h(r) = C_1 + C_2 \log r.$$

Chunk of an annulus



Boundary value problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad (r_1<r<r_2,\ 0<\theta<L),$$

$$u(r,0)=u(r,L)=0 \qquad (r_1<r<r_2),$$

$$u(r_1,\theta)=0,\ \ u(r_2,\theta)=f(\theta) \qquad (0<\theta<L).$$

It is assumed that $r_1>0$, $L<2\pi$.
If $r_1=0$ then the chunk (annular sector) becomes a wedge (circular sector).

We are looking for a solution $u(r,\theta)=h(r)\phi(\theta)$ to Laplace's equation that satisfies the three homogeneous boundary conditions.

PDE holds if

$$r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} = \lambda h,$$

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi.$$

for the same constant λ.

Boundary conditions $u(r,0) = u(r,L) = 0$ hold if

$$\phi(0) = \phi(L) = 0.$$

Boundary condition $u(r_1,\theta) = 0$ holds if

$$h(r_1) = 0.$$

Eigenvalue problem: $\phi'' = -\lambda\phi, \;\; \phi(0) = \phi(L) = 0.$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2,\; n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(\theta) = \sin\frac{n\pi\theta}{L}.$

Dependence on r:

$$r^2h'' + rh' = \lambda h, \qquad h(r_1) = 0.$$

$$\Longrightarrow \;\; h(r) = C_0\Big((\tfrac{r}{r_1})^p - (\tfrac{r_1}{r})^p\Big) \;\; (p = \sqrt{\lambda})$$

Solution of Laplace's equation:

$$u(r,\theta) = \Big((\tfrac{r}{r_1})^{n\pi/L} - (\tfrac{r_1}{r})^{n\pi/L}\Big)\sin\frac{n\pi\theta}{L}, \qquad n = 1, 2, \ldots$$

We are looking for the solution of the reduced boundary value problem as a superposition of solutions with separated variables.

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n \left(\left(\frac{r}{r_1}\right)^{n\pi/L} - \left(\frac{r_1}{r}\right)^{n\pi/L} \right) \sin \frac{n\pi\theta}{L}$$

Boundary condition $u(r_2,\theta) = f(\theta)$ is satisfied if

$$f(\theta) = \sum_{n=1}^{\infty} C_n \left(\left(\frac{r_2}{r_1}\right)^{n\pi/L} - \left(\frac{r_1}{r_2}\right)^{n\pi/L} \right) \sin \frac{n\pi\theta}{L}$$

How do we solve the boundary value problem?

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\,\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad (r_1 < r < r_2,\ 0 < \theta < L),$$

$$u(r_2, \theta) = f(\theta),\ \ u(r, 0) = u(r, L) = u(r_1, \theta) = 0.$$

- Expand f into the Fourier sine series:

$$f(\theta) = \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi\theta}{L}.$$

- Write the solution:

$$u(r, \theta) = \sum_{n=1}^{\infty} C_n\Big(\big(\tfrac{r}{r_1}\big)^{n\pi/L} - \big(\tfrac{r_1}{r}\big)^{n\pi/L}\Big) \sin \tfrac{n\pi\theta}{L},$$

where $C_n = \dfrac{a_n}{\left(\frac{r_2}{r_1}\right)^{n\pi/L} - \left(\frac{r_1}{r_2}\right)^{n\pi/L}}.$