Math 412-501

Theory of Partial Differential Equations

Lecture 2-5: Laplace's equation in polar coordinates (continued). Heat conduction in a rectangle.

Laplace's equation

In Cartesian coordinates (x, y),

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

In polar coordinates (r, θ),

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

or

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Separation of variables: $u(r,\theta) = h(r)\phi(\theta)$.

$$r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} = \lambda h,$$

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi.$$

Proposition Suppose h and ϕ are solutions of the above ODEs for the same value of λ. Then $u(r,\theta) = h(r)\phi(\theta)$ is a solution of Laplace's equation.

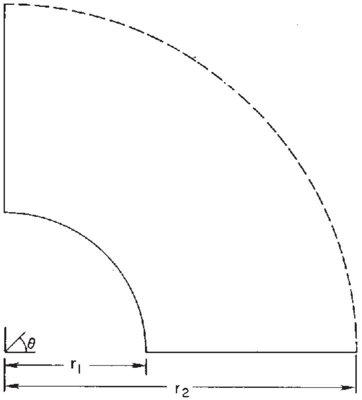
Euler's (or equidimensional) equation

$$r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} - \lambda h = 0 \quad (r > 0)$$

$$\lambda > 0 \implies h(r) = C_1 r^p + C_2 r^{-p} \quad (\lambda = p^2,\ p > 0)$$

$$\lambda = 0 \implies h(r) = C_1 + C_2 \log r$$

Chunk of an annulus



Boundary value problem (annular sector)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad (r_1 < r < r_2,\ 0 < \theta < L),$$

$$u(r,0) = u(r,L) = 0 \qquad (r_1 < r < r_2),$$

$$u(r_1,\theta) = 0,\ \ u(r_2,\theta) = f(\theta) \qquad (0 < \theta < L).$$

It is assumed that $r_1 > 0$, $L < 2\pi$.
If $r_1 = 0$ then the chunk (annular sector) becomes a wedge (circular sector).

We are looking for a solution $u(r,\theta) = h(r)\phi(\theta)$ to Laplace's equation that satisfies the three homogeneous boundary conditions.

PDE holds if

$$r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} = \lambda h,$$

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi.$$

for the same constant λ.

Boundary conditions $u(r, 0) = u(r, L) = 0$ hold if

$$\phi(0) = \phi(L) = 0.$$

Boundary condition $u(r_1, \theta) = 0$ holds if

$$h(r_1) = 0.$$

Eigenvalue problem: $\phi'' = -\lambda\phi, \;\; \phi(0) = \phi(L) = 0.$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(\theta) = \sin\frac{n\pi\theta}{L}$.

Dependence on r:

$$r^2 h'' + rh' = \lambda h, \qquad h(r_1) = 0.$$

$$\Longrightarrow \; h(r) = C_0\Big((\tfrac{r}{r_1})^p - (\tfrac{r_1}{r})^p\Big) \;\; (p = \sqrt{\lambda})$$

Solution of Laplace's equation:

$$u(r,\theta) = \Big((\tfrac{r}{r_1})^{n\pi/L} - (\tfrac{r_1}{r})^{n\pi/L}\Big)\sin\frac{n\pi\theta}{L}, \qquad n = 1, 2, \ldots$$

We are looking for the solution of the reduced boundary value problem as a superposition of solutions with separated variables.

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n \left(\left(\frac{r}{r_1}\right)^{n\pi/L} - \left(\frac{r_1}{r}\right)^{n\pi/L} \right) \sin \frac{n\pi\theta}{L}$$

Boundary condition $u(r_2, \theta) = f(\theta)$ is satisfied if

$$f(\theta) = \sum_{n=1}^{\infty} C_n \left(\left(\frac{r_2}{r_1}\right)^{n\pi/L} - \left(\frac{r_1}{r_2}\right)^{n\pi/L} \right) \sin \frac{n\pi\theta}{L}$$

How do we solve the boundary value problem?

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\,\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad (r_1 < r < r_2,\ 0 < \theta < L),$$

$$u(r_2,\theta) = f(\theta),\ \ u(r,0) = u(r,L) = u(r_1,\theta) = 0.$$

- Expand f into the Fourier sine series:

$$f(\theta) = \sum\nolimits_{n=1}^{\infty} a_n \sin\tfrac{n\pi\theta}{L}.$$

- Write the solution:

$$u(r,\theta) = \sum\nolimits_{n=1}^{\infty} C_n\Big(\big(\tfrac{r}{r_1}\big)^{n\pi/L} - \big(\tfrac{r_1}{r}\big)^{n\pi/L}\Big)\sin\tfrac{n\pi\theta}{L},$$

where $C_n = \dfrac{a_n}{\left(\frac{r_2}{r_1}\right)^{n\pi/L} - \left(\frac{r_1}{r_2}\right)^{n\pi/L}}.$

Boundary value problem (circular sector)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad (0<r<R,\ 0<\theta<L),$$

$$u(r,0)=u(r,L)=0 \qquad (0<r<R),$$

$$u(0,\theta)=0,\ \ u(R,\theta)=f(\theta) \qquad (0<\theta<L).$$

It is assumed that $L<2\pi$.

We are looking for a solution $u(r,\theta)=h(r)\phi(\theta)$ to Laplace's equation that satisfies the three homogeneous boundary conditions.

PDE holds if

$$r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} = \lambda h,$$

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi.$$

for the same constant λ.

Boundary conditions $u(r, 0) = u(r, L) = 0$ hold if

$$\phi(0) = \phi(L) = 0.$$

Boundary condition $u(0, \theta) = 0$ holds if

$$h(0) = 0.$$

Eigenvalue problem: $\phi'' = -\lambda\phi$, $\phi(0) = \phi(L) = 0$.

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(\theta) = \sin\frac{n\pi\theta}{L}$.

Dependence on r:

$$r^2 h'' + rh' = \lambda h, \qquad h(0) = 0.$$

$$\implies h(r) = C_0 r^p \quad (p = \sqrt{\lambda})$$

Solution of Laplace's equation:

$$u(r,\theta) = r^{n\pi/L} \sin\frac{n\pi\theta}{L}, \qquad n = 1, 2, \ldots$$

We are looking for the solution of the reduced boundary value problem as a superposition of solutions with separated variables.

$$u(r,\theta) = \sum\nolimits_{n=1}^{\infty} C_n r^{n\pi/L} \sin \frac{n\pi\theta}{L}$$

Boundary condition $u(R,\theta) = f(\theta)$ is satisfied if

$$f(\theta) = \sum\nolimits_{n=1}^{\infty} C_n R^{n\pi/L} \sin \frac{n\pi\theta}{L}$$

How do we solve the boundary value problem?

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\,\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad (0 < r < R,\ 0 < \theta < L),$$

$$u(R,\theta) = f(\theta),\ \ u(r,0) = u(r,L) = u(0,\theta) = 0.$$

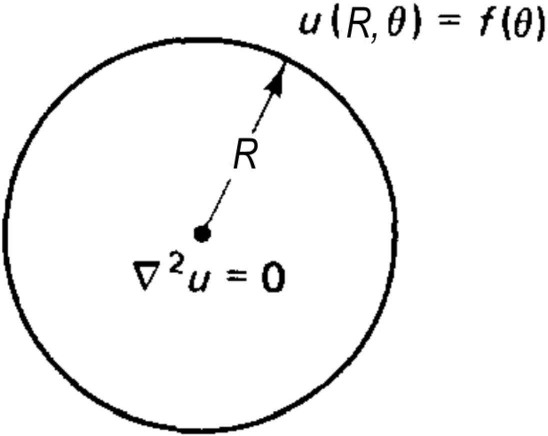
- Expand f into the Fourier sine series:

$$f(\theta) = \sum\nolimits_{n=1}^{\infty} a_n \sin\frac{n\pi\theta}{L}.$$

- Write the solution:

$$u(r,\theta) = \sum\nolimits_{n=1}^{\infty} a_n \Big(\frac{r}{R}\Big)^{n\pi/L} \sin\frac{n\pi\theta}{L}.$$

Circle



Boundary value problem (circle)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad (0<r<R,\ -\pi<\theta<\pi)$$

$$u(R,\theta)=f(\theta) \qquad (-\pi<\theta<\pi)$$

$$u(r,-\pi)=u(r,\pi),\ \ \tfrac{\partial u}{\partial \theta}(r,-\pi)=\tfrac{\partial u}{\partial \theta}(r,\pi)$$

$(0<r<R)$ **(periodic conditions)**

We also need a condition on $u(0,\theta)$.
This is a **singular** condition: $|u(0,\theta)|<\infty$.

Separation of variables: $u(r, \theta) = h(r)\phi(\theta)$.

PDE holds if

$$r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} = \lambda h,$$
$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi.$$

for the same constant λ.

Boundary conditions $u(r, -\pi) = u(r, \pi)$ and $\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$ hold if

$$\phi(-\pi) = \phi(\pi), \quad \phi'(-\pi) = \phi'(\pi).$$

Boundary condition $|u(0, \theta)| < \infty$ holds if

$$|h(0)| < \infty.$$

Eigenvalue problem:

$$\phi'' = -\lambda\phi, \quad \phi(-\pi) = \phi(\pi),\ \phi'(-\pi) = \phi'(\pi).$$

Eigenvalues: $\lambda_n = n^2$, $n = 0, 1, 2, \dots$

Eigenfunctions: $\phi_0 = 1$, $\phi_n(\theta) = \cos n\theta$ and $\psi_n(\theta) = \sin n\theta$, $n = 1, 2, \dots$

Dependence on r:

$$r^2 h'' + rh' = \lambda h, \qquad |h(0)| < \infty.$$

$\lambda > 0 \implies h(r) = C_0 r^p \quad (p = \sqrt{\lambda})$
$\lambda = 0 \implies h(r) = C_0$

Solution of Laplace's equation: $u(r, \theta) = C_0$ or

$$u(r, \theta) = r^n (C_1 \cos n\theta + C_2 \sin n\theta), \qquad n = 1, 2, \dots$$

We are looking for the solution of the reduced boundary value problem as a superposition of solutions with separated variables.

$$u(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta)$$

Boundary condition $u(R,\theta) = f(\theta)$ is satisfied if

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} R^n (C_n \cos n\theta + D_n \sin n\theta)$$

How do we solve the boundary value problem?

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0 \quad (0<r<R,\ -\pi<\theta<\pi),$$

$$u(r,-\pi)=u(r,\pi),\ \ \tfrac{\partial u}{\partial\theta}(r,-\pi)=\tfrac{\partial u}{\partial\theta}(r,\pi),$$
$$u(R,\theta)=f(\theta),\ \ |u(0,\theta)|<\infty.$$

- Expand f into the Fourier series on $[-\pi,\pi]$:

$$f(\theta)=a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta).$$

- Write the solution:

$$u(r,\theta)=a_0+\sum_{n=1}^{\infty}\Big(\frac{r}{R}\Big)^n(a_n\cos n\theta+b_n\sin n\theta).$$

Heat conduction in a rectangle

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \qquad (0 < x < L,\ 0 < y < H),$$

$$u(x, y, 0) = f(x, y) \qquad (0 < x < L,\ 0 < y < H),$$

$$u(0, y, t) = u(L, y, t) = 0 \qquad (0 < y < H),$$

$$u(x, 0, t) = u(x, H, t) = 0 \qquad (0 < x < L).$$

We are looking for solutions to the boundary value problem with separated variables.

Separation of variables: $u(x,y,t) = \phi(x)h(y)G(t)$.
Substitute this into the heat equation:

$$\phi(x)h(y)\frac{dG}{dt} = k\left(\frac{d^2\phi}{dx^2}h(y)G(t) + \phi(x)\frac{d^2h}{dy^2}G(t)\right).$$

Divide both sides by
$k \cdot \phi(x)h(y)G(t) = k \cdot u(x,y,t)$:

$$\frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} + \frac{1}{h} \cdot \frac{d^2h}{dy^2}.$$

It follows that $\frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\lambda$, $\frac{1}{h} \cdot \frac{d^2h}{dy^2} = -\mu$,
$\frac{1}{kG} \cdot \frac{dG}{dt} = -\lambda - \mu$, where λ and μ are **separation constants**.

The variables have been separated:

$$\frac{dG}{dt} = -(\lambda + \mu)kG,$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \frac{d^2h}{dy^2} = -\mu h.$$

Proposition Suppose G, ϕ, and h are solutions of the above ODEs for the same values of λ and μ. Then $u(x, y, t) = \phi(x)h(y)G(t)$ is a solution of the heat equation.

Boundary conditions $u(0, y, t) = u(L, y, t) = 0$ hold if $\phi(0) = \phi(L) = 0$.

Boundary conditions $u(x, 0, t) = u(x, H, t) = 0$ hold if $h(0) = h(H) = 0$.