Math 412-501

Theory of Partial Differential Equations

Lecture 3-10: Applications of Fourier transforms (continued).

Fourier transform

Given a function $h : \mathbb{R} \to \mathbb{C}$, the function

$$\hat{h}(\omega) = \mathcal{F}[h](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x) e^{-i\omega x} \, dx, \qquad \omega \in \mathbb{R}$$

is called the **Fourier transform** of h.

Given a function $H : \mathbb{R} \to \mathbb{C}$, the function

$$\check{H}(x) = \mathcal{F}^{-1}[H](x) = \int_{-\infty}^{\infty} H(\omega) e^{i\omega x} \, d\omega, \qquad x \in \mathbb{R}$$

is called the **inverse Fourier transform** of H.

Initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty),$$
$$u(x,0) = f(x).$$

Solution: $\quad u(x,t) = \int_{-\infty}^{\infty} G(x,\tilde{x},t)\, f(\tilde{x})\, d\tilde{x},$

where $\;G(x,\tilde{x},t) = \dfrac{1}{\sqrt{4\pi kt}}\, e^{-\frac{(x-\tilde{x})^2}{4kt}}.$

The solution is in the integral operator form. The function G is called the **kernel** of the operator. Also, $G(x,\tilde{x},t)$ is called **Green's function** of the problem.

Wave equation on an infinite interval

Initial value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty),$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

We assume that f, g are smooth and rapidly decaying as $x \to \infty$. We search for a solution with the same properties.

Apply the Fourier transform (relative to x) to both sides of the equation:

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial t^2}\right] = c^2\,\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right].$$

Let $U = \mathcal{F}[u]$. That is,

$$U(\omega, t) = \mathcal{F}[u(\cdot, t)](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x}\, dx.$$

Then $\mathcal{F}\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2 U}{\partial t^2}, \quad \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (i\omega)^2 U(\omega, t).$

Hence $\frac{\partial^2 U}{\partial t^2} = c^2 (i\omega)^2 U(\omega, t) = -c^2 \omega^2 U(\omega, t).$

General solution: $U(\omega, t) = a \cos c\omega t + b \sin c\omega t$ $(\omega \neq 0)$, where $a = a(\omega)$, $b = b(\omega)$.

Apply the Fourier transform to the initial conditions:

$$U(\omega, 0) = \hat{f}(\omega), \quad \frac{\partial U}{\partial t}(\omega, 0) = \hat{g}(\omega).$$

Therefore $U(\omega, t) = \hat{f}(\omega)\cos c\omega t + \hat{g}(\omega)\dfrac{\sin c\omega t}{c\omega}$.

We know that

$$\widehat{\chi_{[-a,a]}}(\omega) = \frac{\sin a\omega}{\pi\omega}, \qquad a > 0.$$

Hence $\mathcal{F}^{-1}\left[\frac{\sin c\omega t}{c\omega}\right] = \frac{\pi}{c}\chi_{[-ct,ct]}$, $t > 0$. Then

$$U(\omega, t) = \tfrac{1}{2}\big((e^{ic\omega t}\hat{f}(\omega) + e^{-ic\omega t}\hat{f}(\omega)\big) + \tfrac{\pi}{c}\hat{g}(\omega)\widehat{\chi_{[-ct,ct]}}(\omega).$$

By the shift theorem and the convolution theorem,

$$u(x, t) = \frac{1}{2}\big(f(x + ct) + f(x - ct)\big) + \frac{1}{2c}g * \chi_{[-ct,ct]}(x).$$

$$
\begin{aligned}
g * \chi_{[-ct,ct]}(x) &= \int_{-\infty}^{\infty} g(\tilde{x})\chi_{[-ct,ct]}(x-\tilde{x})\,d\tilde{x} \\
&= \int_{x-ct}^{x+ct} g(\tilde{x})\,d\tilde{x}.
\end{aligned}
$$

Initial value problem:

$$
\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty),
$$

$$
u(x,0) = f(x), \qquad \tfrac{\partial u}{\partial t}(x,0) = g(x).
$$

Solution:

$$
u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tilde{x})\,d\tilde{x}.
$$

Sine and cosine transforms of derivatives

Sine transform: $S[f](\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx$

Cosine transform: $C[f](\omega) = \frac{2}{\pi} \int_0^\infty f(x) \cos \omega x \, dx$

Assume that f and f' are continuous and absolutely integrable on $[0, \infty)$. Then $f(x) \to 0$ as $x \to \infty$. Hence

$$S[f'](\omega) = \frac{2}{\pi} \int_0^\infty f'(x) \sin \omega x \, dx$$

$$= \frac{2}{\pi} f(x) \sin \omega x \, \Big|_{x=0}^{\infty} - \frac{2}{\pi} \int_0^\infty f(x) (\sin \omega x)' \, dx$$

$$= -\omega \, C[f](\omega).$$

Likewise, $C[f'](\omega) = \frac{2}{\pi}\int_0^\infty f'(x)\cos\omega x\,dx$

$$= \frac{2}{\pi} f(x)\cos\omega x \Big|_{x=0}^{\infty} - \frac{2}{\pi}\int_0^\infty f(x)(\cos\omega x)'\,dx$$

$$= -\frac{2}{\pi} f(0) + \omega\, S[f](\omega).$$

$$\boxed{S[f'](\omega) = -\omega\, C[f](\omega)}$$

$$\boxed{C[f'](\omega) = -\frac{2}{\pi} f(0) + \omega\, S[f](\omega)}$$

Now assume that f, f', f'' are continuous and absolutely integrable on $[0, \infty)$. By the above,

$$S[f''](\omega) = -\omega\, C[f'](\omega) = \frac{2}{\pi} f(0)\omega - \omega^2 S[f](\omega),$$

$$C[f''](\omega) = -\frac{2}{\pi} f'(0) + \omega\, S[f'](\omega) = -\frac{2}{\pi} f'(0) - \omega^2 C[f](\omega).$$

$$\boxed{S[f''](\omega) = \frac{2}{\pi} f(0)\omega - \omega^2 S[f](\omega)}$$

$$\boxed{C[f''](\omega) = -\frac{2}{\pi} f'(0) - \omega^2 C[f](\omega)}$$

Fourier transform: $\mathcal{F}[f](\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$

Sine transform: $S[f](\omega) = \dfrac{2}{\pi}\displaystyle\int_{0}^{\infty} f(x)\sin\omega x\,dx$

Cosine transform: $C[f](\omega) = \dfrac{2}{\pi}\displaystyle\int_{0}^{\infty} f(x)\cos\omega x\,dx$

Proposition Suppose that $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$.

(i) If f is even, $f(-x) = f(x)$, then $\mathcal{F}[f]$ is also even; moreover, $C[f](\omega) = 2\mathcal{F}[f](\omega)$ for all $\omega > 0$.

(ii) If f is odd, $f(-x) = -f(x)$, then $\mathcal{F}[f]$ is also odd; moreover, $S[f](\omega) = 2i\,\mathcal{F}[f](\omega)$ for all $\omega > 0$.

Heat equation on a semi-infinite interval

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty),$$

$$\frac{\partial u}{\partial x}(0, t) = 0,$$

$$u(x, 0) = f(x).$$

We search for a solution which is smooth and rapidly decaying as $x \to \infty$. Apply the cosine transform (relative to x) to both sides of the equation:

$$C\left[\frac{\partial u}{\partial t}\right] = k\, C\left[\frac{\partial^2 u}{\partial x^2}\right].$$

Let $U(\omega, t) = C[u](\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty u(x, t) \cos \omega x \, dx.$

Then $C\left[\dfrac{\partial u}{\partial t}\right] = \dfrac{\partial U}{\partial t},$

$$C\left[\frac{\partial^2 u}{\partial x^2}\right] = -\omega^2 U(\omega, t) - \frac{2}{\pi}\frac{\partial u}{\partial x}(0, t) = -\omega^2 U(\omega, t).$$

Hence $\dfrac{\partial U}{\partial t} = -k\omega^2 U(\omega, t).$

General solution: $U(\omega, t) = ce^{-\omega^2 kt}$, where $c = c(\omega)$.

Initial condition $u(x, 0) = f(x)$ implies that
$U(\omega, 0) = C[f](\omega)$.

Therefore $U(\omega, t) = C[f](\omega)\, e^{-\omega^2 kt}$.

Solution: $u(x,t) = \int_0^\infty c(\omega) e^{-\omega^2 kt} \cos \omega x \, d\omega,$

where $c(\omega) = \frac{2}{\pi} \int_0^\infty f(\tilde{x}) \cos \omega \tilde{x} \, d\tilde{x}.$

The same solution can be obtained by separation of variables. The solution can be rewritten in the integral operator form:

$$u(x,t) = \int_0^\infty G(x,\tilde{x},t) f(\tilde{x}) \, d\tilde{x},$$

where $G(x,\tilde{x},t) = \frac{2}{\pi} \int_0^\infty e^{-\omega^2 kt} \cos \omega x \cos \omega \tilde{x} \, d\omega.$

Green's function $G(x, \tilde{x}, t) =$

$$= \frac{1}{\pi} \int_0^\infty e^{-\omega^2 kt} \left(\cos{(x - \tilde{x})\omega} + \cos{(x + \tilde{x})\omega} \right) d\omega$$

We know that

$$\frac{1}{\pi} \int_0^\infty e^{-\alpha\omega^2} \cos \omega y \, d\omega = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{y^2}{4\alpha}}, \quad \alpha > 0.$$

It follows that

$$G(x, \tilde{x}, t) = \frac{1}{\sqrt{4\pi kt}} \left(e^{-\frac{(x-\tilde{x})^2}{4kt}} + e^{-\frac{(x+\tilde{x})^2}{4kt}} \right).$$

Laplace's equation in a half-plane

Boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (-\infty < x < \infty, \;\; 0 < y < \infty),$$

$$u(x,0) = f(x).$$

We assume that f is smooth and rapidly decaying at infinity. We search for a solution with the same properties.

Apply the Fourier transform $\mathcal{F}_x$ (relative to x) to both sides of the equation:

$$\mathcal{F}_x\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{F}_x\left[\frac{\partial^2 u}{\partial y^2}\right] = 0.$$

Let $U(\omega, y) = \mathcal{F}_x[u](\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} u(x, y)e^{-i\omega x}\, dx.$

Then $\mathcal{F}_x\left[\dfrac{\partial^2 u}{\partial y^2}\right] = \dfrac{\partial^2 U}{\partial y^2}, \quad \mathcal{F}_x\left[\dfrac{\partial^2 u}{\partial x^2}\right] = (i\omega)^2 U(\omega, y).$

Hence $\dfrac{\partial^2 U}{\partial y^2} = -(i\omega)^2 U(\omega, y) = \omega^2 U(\omega, y).$

General solution: $U(\omega, y) = ae^{\omega y} + be^{-\omega y} \quad (\omega \neq 0),$

where $a = a(\omega)$, $b = b(\omega)$.

Initial condition $u(x, 0) = f(x)$ implies that
$U(\omega, 0) = \hat{f}(\omega).$

Also, we have a boundary condition $\lim\limits_{y \to \infty} U(\omega, y) = 0.$

Since $U(\omega, y) \to 0$ as $y \to \infty$, it follows that

$$U(\omega, y) = \begin{cases} b(\omega)e^{-\omega y} & \text{if } \omega > 0, \\ a(\omega)e^{\omega y} & \text{if } \omega < 0. \end{cases}$$

Since $U(\omega, 0) = \hat{f}(\omega)$, it follows that

$$U(\omega, y) = \hat{f}(\omega)e^{-y|\omega|}.$$

It turns out that $\mathcal{F}^{-1}[e^{-\alpha|\omega|}](x) = \dfrac{2\alpha}{x^2 + \alpha^2}$, $\alpha > 0$.

Hence $U(\omega, y) = \hat{f}(\omega)\hat{g}(\omega, y)$, where

$g(x, y) = \dfrac{2y}{x^2 + y^2}$.

By the convolution theorem, $u(x, y) = (2\pi)^{-1} f * g$.

Boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (-\infty < x < \infty,\ \ 0 < y < \infty),$$

$$u(x,0) = f(x).$$

Solution:

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x-\tilde{x},y) f(\tilde{x})\,d\tilde{x}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\,\frac{y}{(x-\tilde{x})^2+y^2}\,d\tilde{x}.$$