Math 412-501

Theory of Partial Differential Equations

Lecture 3-11: Review for Exam 3.

Wave equation in polar coordinates

Initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad \text{in} \quad D,$$

$$u\big|_{t=0} = f, \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = g,$$

$$\frac{\partial u}{\partial n}\Big|_{\partial D} = 0,$$

in a domain $D = \{(r, \theta) : \; 0 < r < a, \; 0 < \theta < \pi/2\}$, a quarter-circle (given in polar coordinates).

Initial conditions:

$$u(r, \theta, 0) = f(r, \theta), \qquad \frac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta).$$

Normal derivative:

$$\frac{\partial u}{\partial n}(a, \theta, t) = \frac{\partial u}{\partial r}(a, \theta, t),$$

$$\frac{\partial u}{\partial n}(r, \pi/2, t) = r^{-1}\frac{\partial u}{\partial \theta}(r, \pi/2, t),$$

$$\frac{\partial u}{\partial n}(r, 0, t) = -r^{-1}\frac{\partial u}{\partial \theta}(r, 0, t).$$

Boundary conditions:

$$\frac{\partial u}{\partial r}(a, \theta, t) = 0, \quad \frac{\partial u}{\partial \theta}(r, 0, t) = \frac{\partial u}{\partial \theta}(r, \pi/2, t) = 0.$$

Also, we will need the singular condition

$$|u(0, \theta, t)| < \infty.$$

First we search for **normal modes**: solutions $u(r, \theta, t) = f(r)h(\theta)G(t)$ of the wave equation that satisfy the boundary conditions.

Note that $\phi(r, \theta) = f(r)h(\theta)$ is going to be an eigenfunction of the Neumann Laplacian in D.

Wave equation in polar coordinates:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right).$$

Substitute $u(r, \theta, t) = f(r)h(\theta)G(t)$ into it:

$$f(r)h(\theta)G''(t) = c^2 \Big(f''(r)h(\theta)G(t) + r^{-1}f'(r)\phi(\theta)G(t) + r^{-2}f(r)h''(\theta)G(t) \Big).$$

Divide both sides by $c^2 f(r)h(\theta)G(t)$:

$$\frac{G''(t)}{c^2 G(t)} = \frac{f''(r)}{f(r)} + \frac{f'(r)}{r\,f(r)} + \frac{h''(\theta)}{r^2 h(\theta)}.$$

It follows that

$$\frac{G''(t)}{c^2 G(t)} = \frac{f''(r)}{f(r)} + \frac{f'(r)}{r\,f(r)} + \frac{h''(\theta)}{r^2 h(\theta)} = -\lambda = \text{const.}$$

Hence $G'' = -\lambda c^2 G$ and

$$f''(r)h(\theta) + r^{-1} f'(r)h(\theta) + r^{-2} f(r)h''(\theta) = -\lambda f(r)h(\theta).$$

The latter equation can be rewritten as

$$\nabla^2 \phi = -\lambda \phi, \text{ where } \phi(r,\theta) = f(r)h(\theta).$$

Divide both sides by $r^{-2}f(r)h(\theta)$:

$$\frac{r^2 f''(r)}{f(r)} + \frac{r\, f'(r)}{f(r)} + \frac{h''(\theta)}{h(\theta)} = -\lambda r^2.$$

It follows that

$$\frac{r^2 f''(r)}{f(r)} + \frac{r\, f'(r)}{f(r)} + \lambda r^2 = -\frac{h''(\theta)}{h(\theta)} = \mu = \text{const.}$$

Hence $h'' = -\mu h$ and

$$r^2 f''(r) + r\, f(r) + (\lambda r^2 - \mu) f(r) = 0.$$

Boundary conditions $\frac{\partial u}{\partial \theta}(r, 0, t) = \frac{\partial u}{\partial \theta}(r, \pi/2, t) = 0$ hold if $h'(0) = h'(\pi/2) = 0$.

Boundary conditions $\frac{\partial u}{\partial r}(a, \theta, t) = 0$ and $|u(0, \theta, t)| < \infty$ hold if $f'(a) = 0$ and $|f(0)| < \infty$.

We obtain two eigenvalue problems:

$$r^2 f'' + rf' + (\lambda r^2 - \mu)f = 0, \quad f'(a) = 0, \ |f(0)| < \infty;$$

$$h'' = -\mu h, \quad h'(0) = h'(\pi/2) = 0.$$

The second problem has eigenvalues $\mu_m = (2m)^2$, $m = 0, 1, 2, \ldots$, and eigenfunctions $h_m(\theta) = \cos 2m\theta$. In particular, $h_0 = 1$.

The first eigenvalue problem:

$$r^2 f'' + rf' + (\lambda r^2 - \nu^2)f = 0, \quad |f(0)| < \infty, \ f'(a) = 0.$$

Here $\nu = \sqrt{\mu_m} = 2m$. First assume that $\lambda > 0$.

New coordinate $z = \sqrt{\lambda} \cdot r$ reduces the equation to Bessel's equation of order ν:

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - \nu^2)f = 0.$$

General solution: $f(z) = c_1 J_\nu(z) + c_2 Y_\nu(z)$, where c_1, c_2 are constants.

Hence $f(r) = c_1 J_\nu(\sqrt{\lambda}\, r) + c_2 Y_\nu(\sqrt{\lambda}\, r)$.

Boundary condition $|f(0)| < \infty$ holds if $c_2 = 0$.

Nonzero solution exists if $J'_\nu(\sqrt{\lambda}\, a) = 0$.

Now consider the case $\lambda = 0$. Here

$$r^2 f'' + rf' - \nu^2 f = 0.$$

General solution: $f(r) = \begin{cases} c_1 r^{\nu} + c_2 r^{-\nu} & \text{if } \nu > 0, \\ c_1 + c_2 \log r & \text{if } \nu = 0, \end{cases}$

where c_1, c_2 are constants.

Boundary condition $|f(0)| < \infty$ holds if $c_2 = 0$.

Nonzero solution exists only for $\nu = 0$.

Thus there are infinitely many eigenvalues $\lambda_{m,1}, \lambda_{m,2}, \ldots,$

where $\sqrt{\lambda_{m,n}}\, a = j'_{\nu,n}$ is the nth positive zero of J'_ν (exception: $j'_{0,1} = 0$).

Corresponding eigenfunctions:

$f_{m,n}(r) = J_\nu(\sqrt{\lambda_{m,n}}\, r)$ (note that $f_{0,1} = 1$).

Dependence on t: $G'' = -\lambda c^2 G$

$$\implies G(t) = \begin{cases} c_1 \cos(\sqrt{\lambda}\, ct) + c_2 \sin(\sqrt{\lambda}\, ct), & \lambda > 0 \\ c_1 + c_2 t, & \lambda = 0 \end{cases}$$

Normal modes:

$$J_{2m}(\sqrt{\lambda_{m,n}}\, r) \cdot \cos 2m\theta \cdot \left\{ \begin{array}{c} \cos(\sqrt{\lambda_{m,n}}\, ct) \\ \sin(\sqrt{\lambda_{m,n}}\, ct) \end{array} \right\}$$

and t.

The solution of the initial-boundary value problem is a superposition of normal modes:

$$u(r,\theta,t) = B_{0,1}t +$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{m,n}J_{2m}(\sqrt{\lambda_{m,n}}\,r)\cos 2m\theta\ \cos(\sqrt{\lambda_{m,n}}\,ct)$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{m,n}J_{2m}(\sqrt{\lambda_{m,n}}\,r)\cos 2m\theta\ \sin(\sqrt{\lambda_{m,n}}\,ct).$$

Initial conditions $u(r,\theta,0) = f(r,\theta)$ and $\frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta)$ imply that

$$u(r,\theta,t) = b_{0,1}t +$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{m,n}J_{2m}(\sqrt{\lambda_{m,n}}\,r)\cos 2m\theta\ \cos(\sqrt{\lambda_{m,n}}\,ct)$$

$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{b_{m,n}}{\sqrt{\lambda_{m,n}}\,c}J_{2m}(\sqrt{\lambda_{m,n}}\,r)\cos 2m\theta\ \sin(\sqrt{\lambda_{m,n}}\,ct),$$

where

$$f(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{m,n}J_{2m}(\sqrt{\lambda_{m,n}}\,r)\cos 2m\theta,$$

$$g(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} b_{m,n}J_{2m}(\sqrt{\lambda_{m,n}}\,r)\cos 2m\theta.$$

In particular, suppose that $f(r,\theta) = 0$, $g(r,\theta) = h(r)\cos 4\theta$.

Then $u(r,\theta,t) =$

$$= \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{\lambda_{2,n}}\, c} J_4(\sqrt{\lambda_{2,n}}\, r) \cos 4\theta \, \sin(\sqrt{\lambda_{2,n}}\, ct),$$

where

$$h(r) = \sum_{n=1}^{\infty} b_n J_4(\sqrt{\lambda_{2,n}}\, r)$$

is the Fourier-Bessel series.

Fourier transforms

Fourier transform: $\mathcal{F}[f](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$

Sine transform: $S[f](\omega) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \sin \omega x\, dx$

Cosine transform: $C[f](\omega) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \cos \omega x\, dx$

Inverse Fourier transforms

Inverse Fourier transform:

$$\mathcal{F}^{-1}[f](\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx$$

Inverse sine transform:

$$S^{-1}[f](\omega) = \int_{0}^{\infty} f(x) \sin \omega x \, dx$$

Inverse cosine transform:

$$C^{-1}[f](\omega) = \int_{0}^{\infty} f(x) \cos \omega x \, dx$$

Laplace's equation in a half-plane

Boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (-\infty < x < \infty, \;\; 0 < y < \infty),$$

$$u(x, 0) = f(x).$$

We assume that f is smooth and rapidly decaying at infinity. We search for a solution with the same properties.

Apply the Fourier transform $\mathcal{F}_x$ (relative to x) to both sides of the equation:

$$\mathcal{F}_x\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{F}_x\left[\frac{\partial^2 u}{\partial y^2}\right] = 0.$$

Let $U(\omega, y) = \mathcal{F}_x[u](\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} u(x, y)e^{-i\omega x}\,dx.$

Then $\mathcal{F}_x\left[\dfrac{\partial^2 u}{\partial y^2}\right] = \dfrac{\partial^2 U}{\partial y^2}, \quad \mathcal{F}_x\left[\dfrac{\partial^2 u}{\partial x^2}\right] = (i\omega)^2 U(\omega, y).$

Hence $\dfrac{\partial^2 U}{\partial y^2} = -(i\omega)^2 U(\omega, y) = \omega^2 U(\omega, y).$

General solution: $U(\omega, y) = ae^{\omega y} + be^{-\omega y} \quad (\omega \neq 0),$
where $a = a(\omega)$, $b = b(\omega)$.

Initial condition $u(x, 0) = f(x)$ implies that
$U(\omega, 0) = \hat{f}(\omega).$

Also, we have a boundary condition $\lim_{y\to\infty} U(\omega, y) = 0.$

Since $U(\omega, y) \to 0$ as $y \to \infty$, it follows that

$$U(\omega, y) = \begin{cases} b(\omega)e^{-\omega y} & \text{if} \ \ \omega > 0, \\ a(\omega)e^{\omega y} & \text{if} \ \ \omega < 0. \end{cases}$$

Since $U(\omega, 0) = \hat{f}(\omega)$, it follows that

$$U(\omega, y) = \hat{f}(\omega)e^{-y|\omega|}.$$

It turns out that $\mathcal{F}^{-1}[e^{-\alpha|\omega|}](x) = \frac{2\alpha}{x^2 + \alpha^2}$, $\alpha > 0$.

Hence $U(\omega, y) = \hat{f}(\omega)\hat{g}(\omega, y)$, where

$g(x, y) = \frac{2y}{x^2 + y^2}$.

By the convolution theorem, $u(x, y) = (2\pi)^{-1} f * g$.

Boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (-\infty < x < \infty,\ \ 0 < y < \infty),$$

$$u(x,0) = f(x).$$

Solution:

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x - \tilde{x}, y) f(\tilde{x})\, d\tilde{x}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(\tilde{x})\, \frac{y}{(x - \tilde{x})^2 + y^2}\, d\tilde{x}.$$

Properties of Fourier transforms

Linearity and Shift Theorem

(i) $\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$ for all $a, b \in \mathbb{C}$.

(ii) If $g(x) = f(x + \alpha)$ then $\hat{g}(\omega) = e^{i\alpha\omega}\hat{f}(\omega)$.

(iii) If $h(x) = e^{i\beta x} f(x)$ then $\hat{h}(\omega) = \hat{f}(\omega - \beta)$.

Convolution Theorem

(i) $\mathcal{F}[f \cdot g] = \mathcal{F}[f] * \mathcal{F}[g]$;

(ii) $\mathcal{F}[f * g] = 2\pi\,\mathcal{F}[f] \cdot \mathcal{F}[g]$.

We know that $\widehat{\chi}_{[-a,a]}(\omega) = \dfrac{\sin a\omega}{\pi\omega}$.

Problem 1. Find the Fourier transform of $\chi_{[0,2a]}$.

Solution. Clearly, $\chi_{[0,2a]}(x) = \chi_{[-a,a]}(x-a)$. By the shift theorem,

$$\widehat{\chi}_{[0,2a]}(\omega) = e^{-ia\omega}\widehat{\chi}_{[-a,a]}(\omega) = e^{-ia\omega}\frac{\sin a\omega}{\pi\omega}.$$

Problem 2. Compute $\dfrac{\sin a\omega}{\pi\omega} * \dfrac{\sin a\omega}{\pi\omega}$.

Solution. By the convolution theorem,

$$\mathcal{F}^{-1}\left[\frac{\sin a\omega}{\pi\omega} * \frac{\sin a\omega}{\pi\omega}\right] = 2\pi\chi^2_{[-a,a]} = 2\pi\chi_{[-a,a]}.$$

Hence

$$\frac{\sin a\omega}{\pi\omega} * \frac{\sin a\omega}{\pi\omega} = \frac{2\sin a\omega}{\omega}.$$