Math 412-501

Theory of Partial Differential Equations

Lecture 3-5:
Wave equation in an arbitrary domain.
Laplace's equation in a cylinder.

Wave equation in an arbitrary domain

Initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big), \quad (x,y) \in D,$$

$$u(x,y,0) = f(x,y), \ \ \tfrac{\partial u}{\partial t}(x,y,0) = g(x,y),$$

$$u\big|_{\partial D} = 0.$$

We search for the solution $u(x,y,t)$ as a superposition of solutions with separated variables that satisfy the boundary condition.

For a general domain, we can only separate the time variable from the others.

Separation of variables: $u(x,y,t) = \phi(x,y)G(t)$.
Substitute this into the wave equation:

$$\phi(x,y)\frac{d^2G}{dt^2} = c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)G(t).$$

Divide both sides by $c^2\phi(x,y)G(t) = c^2u(x,y,t)$:

$$\frac{1}{c^2G}\cdot\frac{d^2G}{dt^2} = \frac{1}{\phi}\cdot\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right).$$

It follows that

$$\frac{1}{c^2G}\cdot\frac{d^2G}{dt^2} = \frac{1}{\phi}\cdot\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = -\lambda,$$

where λ is a separation constant.

The time variable has been separated:

$$\frac{d^2 G}{dt^2} = -\lambda c^2 G, \qquad \nabla^2 \phi = -\lambda \phi.$$

Boundary condition $u|_{\partial D} = 0$ holds if $\phi|_{\partial D} = 0$.

Let $\lambda_1 < \lambda_2 \leq \dots$ be eigenvalues of the negative Dirichlet Laplacian in D (counting with multiplicities), and $\phi_1, \phi_2, \dots$ be the corresponding (orthogonal) eigenfunctions.

Dependence on t (assuming $\lambda > 0$):

$$G(t) = C_1 \cos(\sqrt{\lambda}\, ct) + C_2 \sin(\sqrt{\lambda}\, ct).$$

Solution with separated variables:

$$u(x, y, t) = \Big(C_1 \cos(\sqrt{\lambda_n}\, ct) + C_2 \sin(\sqrt{\lambda_n}\, ct)\Big)\phi_n(x, y).$$

This is a **normal mode** of oscillation.

Natural frequency is a frequency of a normal mode.

Natural frequencies: $\omega_n = \dfrac{c\sqrt{\lambda_n}}{2\pi},\ n = 1, 2, \ldots$

We are looking for the solution of the initial-boundary value problem as a superposition of solutions with separated variables.

$$u(x, y, t) =$$
$$= \sum_{n=1}^{\infty} \Big(A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct) \Big) \phi_n(x, y)$$

Substitute the series into the initial conditions:

$$f(x, y) = \sum_{n=1}^{\infty} A_n \phi_n(x, y),$$

$$g(x, y) = \sum_{n=1}^{\infty} B_n \sqrt{\lambda_n}\, c \phi_n(x, y).$$

By construction, eigenfunctions ϕ_n are orthogonal relative to the inner product

$$\langle F, G\rangle = \iint_D F(x,y)\overline{G(x,y)}\,dx\,dy.$$

Moreover, they form a basis of the Hilbert space $L_2(D)$. Hence the expansion of $f(x,y)$ and $g(x,y)$ is possible.

$$\langle f, \phi_n\rangle = A_n\langle \phi_n, \phi_n\rangle \implies A_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}$$

$$\langle g, \phi_n\rangle = B_n\sqrt{\lambda_n}\,c\langle \phi_n, \phi_n\rangle \implies B_n = \frac{\langle g, \phi_n\rangle}{c\sqrt{\lambda_n}\langle \phi_n, \phi_n\rangle}$$

Initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big), \quad (x, y) \in D,$$

$$u(x, y, 0) = f(x, y), \ \ \tfrac{\partial u}{\partial t}(x, y, 0) = g(x, y),$$

$$u\big|_{\partial D} = 0.$$

Solution: $u(x, y, t) =$

$$= \sum_{n=1}^{\infty}\Big(a_n \cos(\sqrt{\lambda_n}\, ct) + \frac{b_n}{c\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}\, ct)\Big)\phi_n(x, y),$$

where $a_n = \dfrac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}$, $b_n = \dfrac{\langle g, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}$.

Vibrating circular membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\nabla^2 u \quad \text{in} \quad D = \{(r,\theta) : r < R\},$$
$$u(r,\theta,0) = f(r,\theta), \;\; \tfrac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta),$$
$$u(R,\theta,t) = 0.$$

Eigenvalues of $-\nabla^2$: $\lambda_{m,n} = (j_{m,n}/R)^2$, where $m = 0,1,2,\dots,$ $n = 1,2,\dots,$ and $j_{m,n}$ is the nth positive zero of the Bessel function J_m.

Eigenfunctions: $\phi_{0,n}(r,\theta) = J_0(j_{0,n}\,r/R)$.

For $m \geq 1$, $\phi_{m,n}(r,\theta) = J_m(j_{m,n}\,r/R)\cos m\theta$ and $\tilde{\phi}_{m,n}(r,\theta) = J_m(j_{m,n}\,r/R)\sin m\theta$.

Equivalently,

$$\phi_{m,n}(r,\theta) = J_m(\sqrt{\lambda_{m,n}}\, r)\cos m\theta, \quad m \geq 0,$$

$$\tilde{\phi}_{m,n}(r,\theta) = J_m(\sqrt{\lambda_{m,n}}\, r)\sin m\theta, \quad m \geq 1.$$

Normal modes:

$$J_m(\sqrt{\lambda_{m,n}}\, r) \cdot \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} \cdot \left\{ \begin{array}{c} \cos(\sqrt{\lambda_{m,n}}\, ct) \\ \sin(\sqrt{\lambda_{m,n}}\, ct) \end{array} \right\}$$

Solution: $u(r, \theta, t) =$

$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n} J_m(\sqrt{\lambda_{m,n}}\, r) \cos m\theta \, \cos(\sqrt{\lambda_{m,n}}\, ct)$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{b_{m,n}}{c\sqrt{\lambda_{m,n}}} J_m(\sqrt{\lambda_{m,n}}\, r) \cos m\theta \, \sin(\sqrt{\lambda_{m,n}}\, ct)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{a}_{m,n} J_m(\sqrt{\lambda_{m,n}}\, r) \sin m\theta \, \cos(\sqrt{\lambda_{m,n}}\, ct)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tilde{b}_{m,n}}{c\sqrt{\lambda_{m,n}}} J_m(\sqrt{\lambda_{m,n}}\, r) \sin m\theta \, \sin(\sqrt{\lambda_{m,n}}\, ct),$$

where

$$f(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{m,n} J_m(\sqrt{\lambda_{m,n}}\, r) \cos m\theta$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{a}_{m,n} J_m(\sqrt{\lambda_{m,n}}\, r) \sin m\theta,$$

$$g(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} b_{m,n} J_m(\sqrt{\lambda_{m,n}}\, r) \cos m\theta$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{b}_{m,n} J_m(\sqrt{\lambda_{m,n}}\, r) \sin m\theta,$$

that is,

$$a_{m,n} = \frac{\langle f, \phi_{m,n} \rangle}{\langle \phi_{m,n}, \phi_{m,n} \rangle}, \qquad \tilde{a}_{m,n} = \frac{\langle f, \tilde{\phi}_{m,n} \rangle}{\langle \tilde{\phi}_{m,n}, \tilde{\phi}_{m,n} \rangle},$$

$$b_{m,n} = \frac{\langle g, \phi_{m,n} \rangle}{\langle \phi_{m,n}, \phi_{m,n} \rangle}, \qquad \tilde{b}_{m,n} = \frac{\langle g, \tilde{\phi}_{m,n} \rangle}{\langle \tilde{\phi}_{m,n}, \tilde{\phi}_{m,n} \rangle}.$$

Note that

$$\langle F, G \rangle = \iint_D F(x,y) \overline{G(x,y)} \, dx \, dy$$

$$= \int_{-\pi}^{\pi} \int_0^R F(r,\theta) \, \overline{G(r,\theta)} \, r \, dr \, d\theta.$$

For example,

$$\langle f, \phi_{m,n} \rangle = \int_{-\pi}^{\pi} \int_0^R f(r,\theta) J_m(\sqrt{\lambda_{m,n}}\, r) \cos m\theta \cdot r\, dr\, d\theta,$$

$$\langle \phi_{m,n}, \phi_{m,n} \rangle = \int_{-\pi}^{\pi} \int_0^R \Big(J_m(\sqrt{\lambda_{m,n}}\, r) \cos m\theta \Big)^2 r\, dr\, d\theta$$

$$= \int_0^R |J_m(\sqrt{\lambda_{m,n}}\, r)|^2 r\, dr \cdot \int_{-\pi}^{\pi} \cos^2 m\theta\, d\theta$$

$$= \pi \int_0^R |J_m(\sqrt{\lambda_{m,n}}\, r)|^2 r\, dr \qquad (m \geq 1)$$

$$= \pi \int_0^R |J_m(j_{m,n}\, r/R)|^2 r\, dr$$

$$= \pi R^2 \int_0^R |J_m(j_{m,n}\, r/R)|^2 (r/R)\, d(r/R)$$

$$= \pi R^2 \int_0^1 |J_m(j_{m,n}\, \rho)|^2 \rho\, d\rho \qquad \textbf{(table integral)}$$

$$= \pi R^2 \cdot \frac{1}{2} |J_m'(j_{m,n})|^2.$$

Let us find the asymptotics of $\langle \phi_{m,n}, \phi_{m,n} \rangle$ as $n \to \infty$.

$$\langle \phi_{m,n}, \phi_{m,n} \rangle = \frac{1}{2}\pi R^2 |J'_m(j_{m,n})|^2$$

$$j_{m,n} = (n + \tfrac{1}{2}m - \tfrac{1}{4})\pi + O(1/n)$$

$$J'_m(z) = -\sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O(z^{-3/2})$$

It follows that

$$J'(j_{m,n}) = -\sqrt{\frac{2}{\pi^2 n}} \sin\left(\pi n - \frac{\pi}{2}\right) + O(n^{-3/2}).$$

Hence $\langle \phi_{m,n}, \phi_{m,n} \rangle = \dfrac{R^2}{\pi n} + O(n^{-2})$ as $n \to \infty$.

Circularly symmetric vibration

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad \text{in} \quad D = \{(r,\theta) : r < R\},$$

$$u(r,\theta,0) = \alpha(r), \;\; \frac{\partial u}{\partial t}(r,\theta,0) = \beta(r),$$

$$u(R,\theta,t) = 0.$$

Since the initial and boundary conditions are circularly symmetric, we expect that the solution does not depend on θ.

This follows from uniqueness of the solution.

Solution: $u(r,\theta,t) =$

$$= \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_{0,n}}\, r)\, \cos(\sqrt{\lambda_{0,n}}\, ct)$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{c\sqrt{\lambda_{0,n}}} J_0(\sqrt{\lambda_{0,n}}\, r)\, \sin(\sqrt{\lambda_{0,n}}\, ct),$$

where

$$a_n = \frac{\langle \alpha, \phi_{0,n}\rangle_r}{\langle \phi_{0,n}, \phi_{0,n}\rangle_r}, \qquad b_n = \frac{\langle \beta, \phi_{0,n}\rangle_r}{\langle \phi_{0,n}, \phi_{0,n}\rangle_r}.$$

Here

$$\langle F, G\rangle_r = \int_0^R F(r)\overline{G(r)}\, r\, dr.$$

Laplace's equation in a circular cylinder

Cylinder $D = \{(x, y, z) : x^2 + y^2 < a^2,\ 0 < z < H\}$.

Circular cylindrical coordinates (r, θ, z):

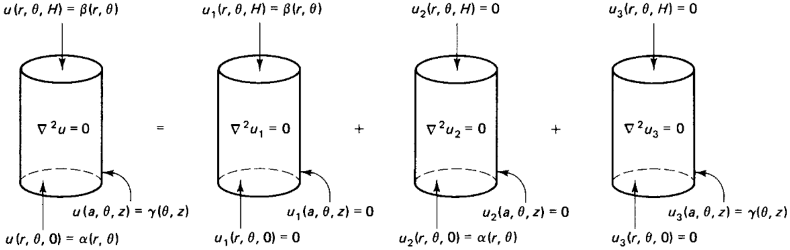
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Laplace's equation in cylindrical coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Boundary conditions:

$$u(r, \theta, H) = \beta(r, \theta) \qquad \text{(top)}$$
$$u(r, \theta, 0) = \alpha(r, \theta) \qquad \text{(bottom)}$$
$$u(a, \theta, z) = \gamma(\theta, z) \qquad \text{(lateral side)}$$



$$u = u_1 + u_2 + u_3,$$

where $\nabla^2 u_1 = \nabla^2 u_2 = \nabla^2 u_3 = 0$,

$$u_1(r,\theta,H) = \beta(r,\theta), \quad u_1(r,\theta,0) = u_1(a,\theta,z) = 0;$$

$$u_2(r,\theta,H) = 0, \quad u_2(r,\theta,0) = \alpha(r,\theta), \quad u_2(a,\theta,z) = 0;$$

$$u_3(r,\theta,H) = u_3(r,\theta,0) = 0, \quad u_3(a,\theta,z) = \gamma(\theta,z).$$

Subproblem 1

$$\nabla^2 u = 0,$$
$$u(r, \theta, H) = \beta(r, \theta),$$
$$u(r, \theta, 0) = 0,$$
$$u(a, \theta, z) = 0.$$

Separation of variables: $u(r, \theta, z) = \phi(r, \theta) h(z)$.
Substitute this into Laplace's equation:

$$\frac{\partial^2 \phi}{\partial r^2} h(z) + \frac{1}{r} \frac{\partial \phi}{\partial r} h(z) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} h(z) + \phi(r, \theta) \frac{d^2 h}{dz^2} = 0,$$

$$\frac{1}{\phi} \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) + \frac{1}{h} \frac{d^2 h}{dz^2} = 0.$$

It follows that

$$\frac{1}{\phi}\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)=-\frac{1}{h}\frac{d^2h}{dz^2}=-\lambda,$$

where λ is a constant.

The variables have been separated:

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=-\lambda\phi, \qquad \frac{d^2h}{dz^2}=\lambda h.$$

Boundary condition $u(a,\theta,z)=0$ holds if $\phi(a,\theta)=0$.

Boundary condition $u(r,\theta,0)=0$ holds if $h(0)=0$.

Eigenvalue problem:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -\lambda\phi, \qquad \phi(a,\theta) = 0.$$

Eigenvalues: $\lambda_{m,n} = (j_{m,n}/a)^2$, $m \geq 0$, $n \geq 1$.

Eigenfunctions:

$$\phi_{m,n}(r,\theta) = J_m(\sqrt{\lambda_{m,n}}\, r)\cos m\theta, \quad m \geq 0,\ n \geq 1,$$
$$\tilde{\phi}_{m,n}(r,\theta) = J_m(\sqrt{\lambda_{m,n}}\, r)\sin m\theta, \quad m \geq 1,\ n \geq 1.$$

Dependence on z:

$$h'' = \lambda h, \quad h(0) = 0 \implies h(z) = c_0 \sinh(\sqrt{\lambda}\, z).$$

Solutions with separated variables:

$$u(r,\theta,z) = \sinh(\sqrt{\lambda_{m,n}}\, z) \cdot \left\{ \begin{array}{c} \phi_{m,n}(r,\theta) \\ \tilde{\phi}_{m,n}(r,\theta) \end{array} \right\}$$

To satisfy the nonhomogeneous boundary condition, we consider a superposition of the above solutions:

$$u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\sinh(\sqrt{\lambda_{m,n}}\,z)\phi_{m,n}(r,\theta)$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{a}_{m,n}\sinh(\sqrt{\lambda_{m,n}}\,z)\tilde{\phi}_{m,n}(r,\theta).$$

Since $u(r,\theta,H)=\beta(r,\theta)$, we have

$$\beta(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\sinh(\sqrt{\lambda_{m,n}}\,H)\phi_{m,n}(r,\theta)$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{a}_{m,n}\sinh(\sqrt{\lambda_{m,n}}\,H)\tilde{\phi}_{m,n}(r,\theta).$$

Coefficients $a_{m,n}$ and $\tilde{a}_{m,n}$ are obtained as before.