Math 412-501

Theory of Partial Differential Equations

Lecture 3-6: Nonhomogeneous problems. Method of eigenfunction expansion.

Nonhomogeneous boundary conditions

Initial-boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L),$$

$$u(0,t) = A,$$

$$u(L,t) = B,$$

$$u(x,0) = f(x) \qquad (0 < x < L).$$

Our goal is to reduce this problem to an analogous one with homogeneous boundary conditions.

Suppose $u_0(x,t)$ is a solution of the equation that satisfies the boundary conditions.

Consider $w(x,t) = u(x,t) - u_0(x,t)$.

u and u_0 are solutions of the heat equation

$\implies$ $w = u - u_0$ is also a solution

u and u_0 satisfy the same boundary conditions

$\implies$ $w = u - u_0$ satisfies homogeneous BCs

$w(x,t)$ is a solution of the following problem:

$$\frac{\partial w}{\partial t} = k\,\frac{\partial^2 w}{\partial x^2} \qquad (0 < x < L),$$

$$w(0,t) = w(L,t) = 0,$$

$$w(x,0) = g(x), \text{ where } g(x) = f(x) - u_0(x,0).$$

How do we find a solution $u_0(x,t)$?

Steady-state (or **equilibrium**) solution:
$u_0(x,t) = u_0(x)$.

$u_0(x)$ is a solution of the boundary value problem

$$\frac{d^2 u_0}{dx^2} = 0, \qquad u_0(0) = A, \;\; u_0(L) = B.$$

The general solution of the equation is
$u_0(x) = c_1 x + c_2$, where c_1, c_2 are constants.

Boundary conditions are satisfied if

$$u_0(x) = A + \frac{B-A}{L}\, x.$$

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L),$$

$$u(0,t) = A, \quad u(L,t) = B,$$

$$u(x,0) = f(x) \qquad (0 < x < L).$$

Solution: $u(x,t) = u_0(x) + w(x,t)$

$$= A + \frac{B-A}{L}\,x + \sum_{n=1}^{\infty} b_n \exp\Big(-(n\pi/L)^2 kt\Big)\sin\frac{n\pi x}{L},$$

where b_n are coefficients of the Fourier sine series on $[0, L]$ of the function $g(x) = f(x) - A - \frac{B-A}{L}x$.

Heat equation with sources:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(x) \qquad (0 < x < L),$$

$$u(0,t) = A, \quad u(L,t) = B,$$

$$u(x,0) = f(x) \qquad (0 < x < L).$$

Suppose $u_0(x,t) = u_0(x)$ is an equilibrium solution.

Then $w(x,t) = u(x,t) - u_0(x)$ is a solution of the homogeneous heat equation with homogeneous boundary conditions.

Solution: $u(x,t) = u_0(x) + w(x,t)$,

where u_0 is the solution of the boundary value problem

$$k\,\frac{d^2u_0}{dx^2} + Q(x) = 0, \qquad u_0(0) = A, \;\; u_0(L) = B;$$

and $w(x,t)$ is the solution of the initial-boundary value problem

$$\frac{\partial w}{\partial t} = k\,\frac{\partial^2 w}{\partial x^2} \qquad (0 < x < L),$$

$$w(0,t) = w(L,t) = 0,$$

$$w(x,0) = f(x) - u_0(x).$$

For some boundary conditions, there is no equilibrium solution. Besides, there is no equilibrium solution if the boundary conditions or the sources depend on time.

Time-dependent problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (0 < x < L),$$

$$u(0,t) = A(t), \quad u(L,t) = B(t) \qquad (t > 0),$$

$$u(x,0) = f(x) \qquad (0 < x < L).$$

We can still reduce the problem to an analogous one with homogeneous boundary conditions.

Suppose $u_0(x,t)$ is a smooth function satisfying the boundary conditions (not necessarily a solution of the PDE). For example,

$$u_0(x,t) = A(t) + \frac{B(t) - A(t)}{L}\, x.$$

Then $w(x,t) = u(x,t) - u_0(x,t)$ satisfies homogeneous boundary conditions.

Substitute $u = u_0 + w$ into the equation:

$$\frac{\partial(u_0 + w)}{\partial t} = k\,\frac{\partial^2(u_0 + w)}{\partial x^2} + Q(x,t),$$

$$\frac{\partial w}{\partial t} = k\,\frac{\partial^2 w}{\partial x^2} + \left(Q(x,t) - \frac{\partial u_0}{\partial t} + k\,\frac{\partial^2 u_0}{\partial x^2}\right).$$

Also, $\; w(x,0) = f(x) - u_0(x,0).$

Solution: $u(x,t) = u_0(x,t) + w(x,t)$,

where

$$u_0(x,t) = A(t) + \frac{B(t) - A(t)}{L}\,x$$

and $w(x,t)$ is the solution of the initial-boundary value problem

$$\frac{\partial w}{\partial t} = k\,\frac{\partial^2 w}{\partial x^2} + \widetilde{Q}(x,t) \qquad (0 < x < L),$$

$$w(0,t) = w(L,t) = 0,$$

$$w(x,0) = g(x) \qquad (0 < x < L),$$

in which $\widetilde{Q}(x,t) = Q(x,t) - \frac{\partial u_0}{\partial t} + k\,\frac{\partial u_0}{\partial x^2}$,
$g(x) = f(x) - u_0(x,0)$.

Method of eigenfunction expansion

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (0 < x < L),$$
$$u(0,t) = u(L,t) = 0,$$
$$u(x,0) = f(x) \qquad (0 < x < L).$$

Consider the related eigenvalue problem

$$\phi'' = -\lambda\phi, \quad \phi(0) = \phi(L) = 0.$$

The eigenvalues are $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \ldots,$ and the corresponding eigenfunctions are $\phi_n(x) = \sin\frac{n\pi x}{L}$.

Any piecewise smooth function on $[0, L]$ can be expanded into the Fourier sine series. In particular,

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$$

where $a_1(t), a_2(t), \ldots$ are some functions.

Let us assume that the series can be differentiated term-by-term (one can show this; an important reason is that $u(x,t)$ satisfies homogeneous boundary conditions). Then

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n'(t)\phi_n(x),$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} a_n(t)\phi_n''(x) = -\sum_{n=1}^{\infty} \lambda_n a_n(t)\phi_n(x).$$

Also,
$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x),$$
$$f(x) = \sum_{n=1}^{\infty} b_n\phi_n(x).$$
Substitute all series into the equation:
$$\sum_{n=1}^{\infty} a_n'(t)\phi_n(x) =$$
$$= -k\sum_{n=1}^{\infty} \lambda_n a_n(t)\phi_n(x) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x),$$
$$\sum_{n=1}^{\infty}\big(a_n'(t) + k\lambda_n a_n(t) - q_n(t)\big)\phi_n(x) = 0.$$
Initial condition is satisfied if
$$\sum_{n=1}^{\infty} a_n(0)\phi_n(x) = \sum_{n=1}^{\infty} b_n\phi_n(x).$$

It follows that for any $n \geq 1$ we have

$$a_n'(t) + k\lambda_n a_n(t) = q_n(t), \qquad a_n(0) = b_n.$$

This is an initial value problem for a_n.

Multiply both sides of the ODE by $e^{\lambda_n kt}$:

$$e^{\lambda_n kt} a_n'(t) + k\lambda_n e^{\lambda_n kt} a_n(t) = e^{\lambda_n kt} q_n(t),$$

$$\left(e^{\lambda_n kt} a_n(t)\right)' = e^{\lambda_n kt} q_n(t),$$

$$e^{\lambda_n kt} a_n(t) - a_n(0) = \int_0^t e^{\lambda_n k\tau} q_n(\tau)\, d\tau,$$

$$a_n(t) = b_n e^{-\lambda_n kt} + e^{-\lambda_n kt} \int_0^t e^{\lambda_n k\tau} q_n(\tau)\, d\tau.$$

Solution: $u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$,

where

$$a_n(t) = b_n e^{-\lambda_n k t} + e^{-\lambda_n k t} \int_0^t e^{\lambda_n k \tau} q_n(\tau)\, d\tau,$$

$$\lambda_n = (n\pi/L)^2,$$

$$b_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{2}{L} \int_0^L f(\xi) \sin \frac{n\pi\xi}{L}\, d\xi,$$

$$q_n(\tau) = \frac{\langle Q(\cdot,\tau), \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{2}{L} \int_0^L Q(\xi,\tau) \sin \frac{n\pi\xi}{L}\, d\xi.$$

If $Q = 0$ then $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n k t} \phi_n(x)$,

just as the separation of variables yields.

Poisson's equation

Poisson's equation is a nonhomogeneous version of Laplace's equation: $\nabla^2 u = Q$.

Consider a boundary value problem:

$$\nabla^2 u = Q \quad \text{in the domain } D,$$
$$u\big|_{\partial D} = \alpha,$$

where Q is a function on D and α is a function on the boundary ∂D.

There are two ways to solve this problem.

Method 1. Suppose u_0 is a smooth function in the domain D such that $\nabla^2 u_0 = Q$.

Then $u = u_0 + w$, where w is the solution of the boundary value problem for Laplace's equation:

$$\nabla^2 u = 0 \quad \text{in the domain } D,$$

$$u\big|_{\partial D} = \beta,$$

where $\beta(\mathbf{x}) = \alpha(\mathbf{x}) - u_0(\mathbf{x})$ for any $\mathbf{x} \in \partial D$.

We know how to solve the latter problem if D is a rectangle or a circle. Method 1 applies only to certain functions Q as it relies on a lucky guess.

Example: $\nabla^2 u = Q(x, y) = e^{-x} \cos 2y - \sin 3x \sin y$.

Lucky guess: $u_0(x, y) = -\frac{1}{3} e^{-x} \cos 2y + \frac{1}{10} \sin 3x \sin y$.

Method 2. $u = u_1 + u_2$,

where u_1 is the solution of the problem

$$\nabla^2 u_1 = Q \quad \text{in the domain } D,$$
$$u_1\big|_{\partial D} = 0,$$

while u_2 is the solution of the problem

$$\nabla^2 u_2 = 0 \quad \text{in the domain } D,$$
$$u_2\big|_{\partial D} = \alpha.$$

We know how to solve the latter problem if D is a rectangle or a circle. The former problem is solved by method of eigenfunction expansion.

Boundary value problem:

$$\nabla^2 u_1 = Q \quad \text{in the domain } D,$$
$$u_1\big|_{\partial D} = 0.$$

Let $\lambda_1 < \lambda_2 \leq \ldots$ be eigenvalues of the negative Dirichlet Laplacian in D (counting with multiplicities), and $\phi_1, \phi_2, \ldots$ be the corresponding (orthogonal) eigenfunctions.

We have that

$$u_1(x,y) = \sum_{n=1}^{\infty} c_n \phi_n(x,y),$$

$$Q(x,y) = \sum_{n=1}^{\infty} q_n \phi_n(x,y).$$

The Laplacian $\nabla^2 u_1$ can be evaluated term-by-term (it follows from the fact that $u_1|_{\partial D} = 0$). Hence

$$\nabla^2 u_1 = \sum_{n=1}^{\infty} c_n \nabla^2 \phi_n = -\sum_{n=1}^{\infty} \lambda_n c_n \phi_n.$$

Thus $-\lambda_n c_n = q_n$ for $n = 1, 2, \ldots$.

Solution:

$$u_1 = -\sum_{n=1}^{\infty} \lambda_n^{-1} q_n \phi_n, \quad \text{where} \quad q_n = \frac{\langle Q, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$