Math 412-501

Theory of Partial Differential Equations

Lecture 3-7: Poisson's equation. Complex form of Fourier series. Fourier transforms.

Poisson's equation

Poisson's equation is the nonhomogeneous version of Laplace's equation: $\nabla^2 u = Q$.

Consider a boundary value problem:

$$\nabla^2 u = Q \quad \text{in the domain } D,$$
$$u\big|_{\partial D} = \alpha,$$

where Q is a function on D and α is a function on the boundary ∂D.

There are two ways to solve this problem.

Method 1. Suppose u_0 is a smooth function in the domain D such that $\nabla^2 u_0 = Q$.

Then $u = u_0 + w$, where w is the solution of the boundary value problem for Laplace's equation:

$$\nabla^2 u = 0 \quad \text{in the domain } D,$$

$$u\big|_{\partial D} = \beta,$$

where $\beta(\mathbf{x}) = \alpha(\mathbf{x}) - u_0(\mathbf{x})$ for any $\mathbf{x} \in \partial D$.

We know how to solve the latter problem if D is a rectangle or a circle. Method 1 applies only to certain functions Q as it relies on a lucky guess.

Example: $\nabla^2 u = Q(x, y) = e^{-x} \cos 2y - \sin 3x \sin y$.

Lucky guess: $u_0(x, y) = -\frac{1}{3} e^{-x} \cos 2y + \frac{1}{10} \sin 3x \sin y$.

Method 2. $u = u_1 + u_2$,

where u_1 is the solution of the problem

$$\nabla^2 u_1 = Q \quad \text{in the domain } D,$$
$$u_1\big|_{\partial D} = 0,$$

while u_2 is the solution of the problem

$$\nabla^2 u_2 = 0 \quad \text{in the domain } D,$$
$$u_2\big|_{\partial D} = \alpha.$$

We know how to solve the latter problem if D is a rectangle or a circle. The former problem is solved by method of eigenfunction expansion.

Boundary value problem:

$$\nabla^2 u_1 = Q \quad \text{in the domain } D,$$
$$u_1\big|_{\partial D} = 0.$$

Let $\lambda_1 < \lambda_2 \leq \dots$ be eigenvalues of the negative Dirichlet Laplacian in D (counting with multiplicities), and $\phi_1, \phi_2, \dots$ be the corresponding (orthogonal) eigenfunctions.

We have that

$$u_1(x,y) = \sum_{n=1}^{\infty} c_n \phi_n(x,y),$$

$$Q(x,y) = \sum_{n=1}^{\infty} q_n \phi_n(x,y).$$

The Laplacian $\nabla^2 u_1$ can be evaluated term-by-term (it follows from the fact that $u_1|_{\partial D} = 0$). Hence

$$\nabla^2 u_1 = \sum_{n=1}^{\infty} c_n \nabla^2 \phi_n = -\sum_{n=1}^{\infty} \lambda_n c_n \phi_n.$$

Thus $-\lambda_n c_n = q_n$ for $n = 1, 2, \dots$.

Solution:

$$u_1 = -\sum_{n=1}^{\infty} \lambda_n^{-1} q_n \phi_n, \quad \text{where} \quad q_n = \frac{\langle Q, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

Heat equation on an infinite interval

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty),$$

$$u(0,t) = 0, \qquad \lim_{x\to\infty} u(x,t) = 0,$$

$$u(x,0) = f(x) \qquad (0 < x < \infty).$$

The problem is supposed to describe heat conduction in a very long rod.

We expect that the solution is the limit of solutions on intervals $[0, L]$ as $L \to \infty$.

The problem on a finite interval:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L),$$

$$u(0,t) = u(L,t) = 0,$$

$$u(x,0) = f(x) \qquad (0 < x < L).$$

Solution: Expand f into the Fourier sine series on $[0, L]$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where $b_n = \frac{2}{L}\int_0^L f(\tilde{x}) \sin \frac{n\pi \tilde{x}}{L}\, d\tilde{x}.$

Then $u(x,t) = \sum_{n=1}^{\infty} b_n \exp\big(-(n\pi/L)^2 kt\big) \sin \frac{n\pi x}{L}.$

For any $\omega > 0$ let $B(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(\tilde{x}) \sin \omega\tilde{x}\, d\tilde{x}$.

For simplicity, assume that $f(x) = 0$ for $x > L_0$. Then $B(\omega)$ is well defined. Moreover,

$$b_n = \frac{2}{L}\int_0^L f(\tilde{x}) \sin \frac{n\pi\tilde{x}}{L}\, d\tilde{x} = \frac{\pi}{L} B\Big(\frac{n\pi}{L}\Big)$$

provided that $L \geq L_0$. Therefore

$$u(x,t) = \frac{\pi}{L}\sum\nolimits_{n=1}^{\infty} B(\omega_n) \exp(-\omega_n^2 kt) \sin \omega_n x,$$

where $\omega_n = n\pi/L$. It follows that

$$\lim_{L\to\infty} u(x,t) = \int_0^\infty B(\omega) e^{-\omega^2 kt} \sin \omega x \, d\omega.$$

The problem on the infinite interval:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty),$$
$$u(0,t) = 0, \qquad \lim_{x\to\infty} u(x,t) = 0,$$
$$u(x,0) = f(x) \qquad (0 < x < \infty).$$

Let us try and solve this problem by separation of variables. First we search for solutions $u(x,t) = \phi(x)G(t)$ of the equation that satisfy the boundary conditions. The PDE holds if

$$\tfrac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \tfrac{dG}{dt} = -\lambda kt,$$

where λ is a separation constant.

Boundary conditions $u(0,t) = u(\infty,t) = 0$ hold if $\phi(0) = \phi(\infty) = 0$.

Eigenvalue problem on $(0, \infty)$:

$$\phi'' = -\lambda\phi, \quad \phi(0) = \phi(\infty) = 0.$$

This problem has no eigenvalues. If we drop the condition $\phi(\infty) = 0$ then any $\lambda \in \mathbb{C}$ will be an eigenvalue, which is bad too.

The right decision is to relax the condition:

$$\phi'' = -\lambda\phi, \quad \phi(0) = 0, \ |\phi(\infty)| < \infty.$$

Eigenvalues: $\lambda = \omega^2$, where $\omega > 0$.

Eigenfunctions: $\phi_\omega(x) = \sin \omega x$.

Dependence on t:

$$G' = -\lambda k G \implies G(t) = c_0 e^{-\lambda k t}$$

Solutions with separated variables:

$$u_\omega(x,t) = e^{-\omega^2 kt} \sin \omega x, \qquad \omega > 0.$$

Now we search for the solution of the initial-boundary value problem as a superposition of solutions with separated variables:

$$u(x,t) = \int_0^\infty B(\omega) e^{-\omega^2 kt} \sin \omega x \, d\omega.$$

The initial condition $u(x,0) = f(x)$ is satisfied if

$$f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega.$$

The right-hand side is called a **Fourier integral**.

Solution: Expand f into the Fourier integral:

$$f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega.$$

Then $u(x,t) = \int_0^\infty B(\omega) e^{-\omega^2 kt} \sin \omega x \, d\omega.$

How do we expand f into the Fourier integral?

Approximation by finite-interval problems suggests that

$$B(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx.$$

Fourier sine transform

Let f be a function on $(0, \infty)$. The function

$$S[f](\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx, \qquad \omega > 0$$

is called the **Fourier sine transform** of f.

The transform $S[f]$ is well defined if the integral converges for all $\omega > 0$.

One sufficient condition is $\int_0^\infty |f(x)| \, dx < \infty$.

Given a function F on $(0, \infty)$, the function

$$S^{-1}[F](x) = \int_0^\infty F(\omega) \sin \omega x \, d\omega, \qquad x > 0$$

is called the **inverse Fourier sine transform** of F.

Theorem Suppose f is an absolutely integrable function on $(0, \infty)$ and let $F = S[f]$ be its Fourier sine transform.

(i) If f is smooth then $f = S^{-1}[F]$.

(ii) If f is piecewise smooth then the inverse Fourier sine transform $S^{-1}[F]$ is equal to f at points of continuity. Otherwise

$$S^{-1}[F](x) = \frac{f(x+) + f(x-)}{2}.$$

Fourier cosine transform

Given a function f on $(0, \infty)$, the function

$$C[f](\omega) = \frac{2}{\pi} \int_0^\infty f(x) \cos \omega x \, dx, \qquad \omega > 0$$

is called the **Fourier cosine transform** of f.

Given a function F on $(0, \infty)$, the function

$$C^{-1}[F](x) = \int_0^\infty F(\omega) \cos \omega x \, d\omega, \qquad x > 0$$

is the **inverse Fourier cosine transform** of F.

Theorem Suppose f is an absolutely integrable function on $(0, \infty)$ and let $F = C[f]$ be its Fourier cosine transform. If f is smooth then $f = C^{-1}[F]$.

A Fourier series on the interval $[-L, L]$:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

A Fourier series in the **complex form**:

$$\sum_{n=-\infty}^{\infty} c_n \exp \frac{in\pi x}{L}.$$

Note that for any $y \in \mathbb{R}$,

$$e^{iy} = \cos y + i \sin y, \qquad e^{-iy} = \cos y - i \sin y,$$

$$\cos y = \tfrac{1}{2}(e^{iy} + e^{-iy}), \qquad \sin y = \tfrac{1}{2i}(e^{iy} - e^{-iy}).$$

Hence both forms of the Fourier series are equivalent. Coefficients are related as follows:

$$a_0 = c_0, \qquad a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n}), \;\; n \geq 1.$$

For any $n \in \mathbb{Z}$, let $\phi_n(x) = e^{in\pi x/L}$. Functions ϕ_n are orthogonal relative to the inner product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)\overline{g(x)}\, dx.$$

Indeed, if $n \neq m$, then

$$\langle \phi_n, \phi_m \rangle = \int_{-L}^{L} e^{in\pi x/L}\, \overline{e^{im\pi x/L}}\, dx$$

$$= \int_{-L}^{L} e^{in\pi x/L}\, e^{-im\pi x/L}\, dx = \int_{-L}^{L} e^{i(n-m)\pi x/L}\, dx$$

$$= \frac{L}{i(n-m)\pi}\, e^{i(n-m)\pi x/L}\, \Big|_{-L}^{L} = 0.$$

Also,

$$\langle \phi_n, \phi_n \rangle = \int_{-L}^{L} |\phi_n(x)|^2 \, dx = \int_{-L}^{L} dx = 2L.$$

Functions ϕ_n form a basis in the Hilbert space $L_2([-L, L])$. Any square-integrable function f on $[-L, L]$ is expanded into a series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

that converges in the mean. Coefficients are obtained as usual:

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} \, dx.$$

Fourier transform

Given a function $f : \mathbb{R} \to \mathbb{C}$, the function

$$\hat{f}(\omega) = \mathcal{F}[f](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx, \qquad \omega \in \mathbb{R}$$

is called the **Fourier transform** of f.

Given a function $F : \mathbb{R} \to \mathbb{C}$, the function

$$\mathcal{F}^{-1}[F](x) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega, \qquad x \in \mathbb{R}$$

is the **inverse Fourier transform** of F.

Theorem Suppose f is an absolutely integrable function on $(-\infty, \infty)$ and let $F = \mathcal{F}[f]$ be its Fourier transform. If f is smooth then $f = \mathcal{F}^{-1}[F]$.