Math 412-501

Theory of Partial Differential Equations

Lecture 3-8:
Properties of Fourier transforms.

Complex form of Fourier series

A Fourier series on the interval $[-L, L]$:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

A Fourier series in the **complex form**:

$$\sum_{n=-\infty}^{\infty} c_n \exp \frac{in\pi x}{L}.$$

For any $y \in \mathbb{R}$,

$$e^{iy} = \cos y + i \sin y, \qquad e^{-iy} = \cos y - i \sin y,$$

$$\cos y = \tfrac{1}{2}(e^{iy} + e^{-iy}), \qquad \sin y = \tfrac{1}{2i}(e^{iy} - e^{-iy}).$$

Hence both forms of the Fourier series are equivalent.

For any $n \in \mathbb{Z}$, let $\phi_n(x) = e^{in\pi x/L}$. Functions ϕ_n are orthogonal relative to the inner product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)\overline{g(x)}\, dx.$$

Indeed, if $n \neq m$, then

$$\langle \phi_n, \phi_m \rangle = \int_{-L}^{L} e^{in\pi x/L}\, \overline{e^{im\pi x/L}}\, dx$$

$$= \int_{-L}^{L} e^{in\pi x/L}\, e^{-im\pi x/L}\, dx = \int_{-L}^{L} e^{i(n-m)\pi x/L}\, dx$$

$$= \frac{L}{i(n-m)\pi}\, e^{i(n-m)\pi x/L}\, \Big|_{-L}^{L} = 0.$$

Also,

$$\langle \phi_n, \phi_n \rangle = \int_{-L}^{L} |\phi_n(x)|^2 \, dx = \int_{-L}^{L} dx = 2L.$$

Functions ϕ_n form a basis in the Hilbert space $L_2([-L, L])$. Any square-integrable function f on $[-L, L]$ is expanded into a series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

that converges in the mean. Coefficients are obtained as usual:

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} \, dx.$$

Fourier transform

Given a function $h : \mathbb{R} \to \mathbb{C}$, the function

$$\hat{h}(\omega) = \mathcal{F}[h](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x) e^{-i\omega x}\, dx, \qquad \omega \in \mathbb{R}$$

is called the **Fourier transform** of h.

Given a function $H : \mathbb{R} \to \mathbb{C}$, the function

$$\check{H}(x) = \mathcal{F}^{-1}[H](x) = \int_{-\infty}^{\infty} H(\omega) e^{i\omega x}\, d\omega, \qquad x \in \mathbb{R}$$

is called the **inverse Fourier transform** of H.

Note that $\mathcal{F}^{-1}[H](x) = 2\pi \cdot \mathcal{F}[H](-x)$.

Discrepancy in the definitions

"Mathematical" notation (used above):

inner product: $\langle f, g\rangle = \int_{-L}^{L} f(x)\overline{g(x)}\,dx;$

Fourier coefficients:

$$c_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx;$$

Fourier transform: $\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx;$

inverse Fourier transform: $\check{F}(x) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega.$

Discrepancy in the definitions

"Physical" notation (used by Haberman):

inner (bra-ket) product: $\langle f | g \rangle = \int_{-L}^{L} \overline{f(x)} g(x)\, dx;$

Fourier coefficients:

$$c_n = \frac{\langle f | \phi_n \rangle}{\langle \phi_n | \phi_n \rangle} = \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L}\, dx;$$

Fourier transform: $\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx;$

inverse Fourier transform: $\check{F}(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega.$

Theorem Suppose h is an absolutely integrable function on $(-\infty, \infty)$ and let $H = \mathcal{F}[h]$ be its Fourier transform.

(i) If h is smooth then $h = \mathcal{F}^{-1}[H]$.

(ii) If h is piecewise smooth then the inverse Fourier transform $\mathcal{F}^{-1}[H]$ is equal to h at points of continuity. Otherwise

$$\mathcal{F}^{-1}[H](x) = \frac{h(x+) + h(x-)}{2}.$$

In particular, any smooth, absolutely integrable function $h : \mathbb{R} \to \mathbb{C}$ is represented as a **Fourier integral**

$$h(x) = \int_{-\infty}^{\infty} H(\omega) e^{i\omega x} \, d\omega.$$

Proposition 1

(i) $\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$ for all $a, b \in \mathbb{C}$.

(ii) If $g(x) = f(x + \alpha)$ then $\hat{g}(\omega) = e^{i\alpha\omega}\hat{f}(\omega)$.

(iii) If $h(x) = e^{i\beta x} f(x)$ then $\hat{h}(\omega) = \hat{f}(\omega - \beta)$.

Proof of (ii): $\hat{g}(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x + \alpha)e^{-i\omega x}\, dx$

$$= \frac{e^{i\alpha\omega}}{2\pi}\int_{\mathbb{R}} f(x + \alpha)e^{-i\omega(x+\alpha)}\, dx$$

$$= \frac{e^{i\alpha\omega}}{2\pi}\int_{\mathbb{R}} f(\tilde{x})e^{-i\omega\tilde{x}}\, d\tilde{x} = e^{i\alpha\omega}\hat{f}(\omega).$$

Example. $f(x) = \begin{cases} 1, & |x| \le a, \\ 0, & |x| > a. \end{cases}$

$$\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \frac{1}{2\pi}\int_{-a}^{a} e^{-i\omega x}\,dx$$

$$= -\frac{1}{2\pi \cdot i\omega}\ e^{-i\omega x}\Big|_{-a}^{a} = \frac{e^{ia\omega} - e^{-ia\omega}}{2\pi \cdot i\omega} = \frac{\sin a\omega}{\pi\omega}, \quad \omega \ne 0.$$

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-a}^{a} dx = \frac{a}{\pi} = \lim_{\omega\to 0}\frac{\sin a\omega}{\pi\omega}.$$

Therefore $$\int_{-\infty}^{\infty}\frac{\sin a\omega}{\pi\omega}e^{i\omega x}\,d\omega = \begin{cases} 1, & |x| \le a, \\ 1/2, & |x| = a, \\ 0, & |x| > a. \end{cases}$$

Proposition 2 Suppose that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.

Then (i) $\hat{f}$ is well defined and bounded;
(ii) $\hat{f}$ is continuous;
(iii) $\hat{f}(\omega) \to 0$ as $\omega \to \infty$.

$$|\hat{f}(\omega)| = \frac{1}{2\pi}\left|\int_{\mathbb{R}} f(x)e^{-i\omega x}\, dx\right| \leq \frac{1}{2\pi}\int_{\mathbb{R}} |f(x)|\, dx$$

Statement (iii) holds if $f = \chi_{[-a,a]}$.

Shift theorem $\implies$ (iii) holds for any $f = \chi_{[a,b]}$.

Linearity $\implies$ (iii) holds for piecewise constant functions.

Finally, for any $\varepsilon > 0$ there exists a piecewise constant function f_ε such that $\int_{-\infty}^{\infty} |f - f_\varepsilon|\, dx < \varepsilon$.

Theorem 1 Let f be a smooth function such that both f and f' are absolutely integrable on $\mathbb{R}$. Then

(i) $\widehat{f'}(\omega) = i\omega \cdot \hat{f}(\omega)$;

(ii) $\hat{f}(\omega) = \alpha(\omega)/\omega$, where $\lim\limits_{\omega\to\infty} \alpha(\omega) = 0$.

Proof of (i): $\widehat{f'}(\omega) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} f'(x) e^{-i\omega x}\, dx$

$$= \frac{1}{2\pi} f(x) e^{-i\omega x} \Big|_{x=-\infty}^{\infty} - \frac{1}{2\pi} \int_{\mathbb{R}} f(x) (e^{-i\omega x})'\, dx$$

$$= \frac{i\omega}{2\pi} \int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx = i\omega \cdot \hat{f}(\omega).$$

f and f' are absolutely integrable $\implies \lim\limits_{x\to\infty} f(x) = 0$

Corollary Let f be a smooth function such that $f, f', f'', \ldots, f^{(k)}$ are all absolutely integrable on $\mathbb{R}$.

Then (i) $\widehat{f^{(k)}}(\omega) = (i\omega)^k \hat{f}(\omega)$;

(ii) $\hat{f}(\omega) = \alpha(\omega)/\omega^k$, where $\lim_{\omega\to\infty} \alpha(\omega) = 0$.

Theorem 2 Let f be a function on $\mathbb{R}$ such that $\int_{\mathbb{R}} (1 + |x|^k)|f(x)|\,dx < \infty$ for some integer $k \geq 1$.

Then (i) $\hat{f}$ is k times differentiable;

(ii) $\hat{f}^{(k)}(\omega) = (-i)^k \mathcal{F}[x^k f(x)](\omega)$.

Convolution

Suppose $f, g : \mathbb{R} \to \mathbb{C}$ are bounded, absolutely integrable functions. The function

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\, dy$$

is called the **convolution** of f and g.

Lemma $f * g = g * f$.

Proof: Let $z = x - y$. Then

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy$$

$$= \int_{-\infty}^{\infty} f(x-z)g(z)\, dz = (g * f)(x).$$

Convolution Theorem

(i) $\mathcal{F}[f \cdot g] = \mathcal{F}[f] * \mathcal{F}[g]$;

(ii) $\mathcal{F}[f * g] = 2\pi\, \mathcal{F}[f] \cdot \mathcal{F}[g]$.

Proof of (ii): $\quad \mathcal{F}[f * g](\omega) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} (f * g)(x) e^{-i\omega x}\, dx$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(x - y) e^{-i\omega x}\, dx\, dy \qquad (x = y + z)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(z) e^{-i\omega(y+z)}\, dz\, dy = 2\pi\, \hat{f}(\omega) \hat{g}(\omega).$$

Plancherel's Theorem (a.k.a. Parseval's Theorem)

(i) If a function f is both absolutely integrable and square-integrable on $\mathbb{R}$, then $\mathcal{F}[f]$ is also square-integrable. Moreover,

$$\int_{\mathbb{R}} |f(x)|^2\,dx = 2\pi \int_{\mathbb{R}} |\hat{f}(\omega)|^2\,d\omega.$$

(ii) If functions f, g are absolutely integrable and square-integrable on $\mathbb{R}$, then

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\,dx = 2\pi \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{g}(\omega)}\,d\omega.$$

That is, $\langle f, g\rangle = 2\pi\,\langle \hat{f}, \hat{g}\rangle$.