Math 412-501

Theory of Partial Differential Equations

Lecture 3-9: Convolution theorem. Applications of Fourier transforms.

Fourier transform

Given a function $h : \mathbb{R} \to \mathbb{C}$, the function

$$\hat{h}(\omega) = \mathcal{F}[h](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x) e^{-i\omega x} \, dx, \qquad \omega \in \mathbb{R}$$

is called the **Fourier transform** of h.

Given a function $H : \mathbb{R} \to \mathbb{C}$, the function

$$\check{H}(x) = \mathcal{F}^{-1}[H](x) = \int_{-\infty}^{\infty} H(\omega) e^{i\omega x} \, d\omega, \qquad x \in \mathbb{R}$$

is called the **inverse Fourier transform** of H.

Theorem Suppose h is an absolutely integrable function on $(-\infty, \infty)$ and let $H = \mathcal{F}[h]$ be its Fourier transform.

(i) If h is smooth then $h = \mathcal{F}^{-1}[H]$.

(ii) If h is piecewise smooth then the inverse Fourier transform $\mathcal{F}^{-1}[H]$ is equal to h at points of continuity. Otherwise

$$\mathcal{F}^{-1}[H](x) = \frac{h(x+) + h(x-)}{2}.$$

In particular, any smooth, absolutely integrable function $h : \mathbb{R} \to \mathbb{C}$ is represented as a **Fourier integral**

$$h(x) = \int_{-\infty}^{\infty} H(\omega) e^{i\omega x} \, d\omega.$$

Proposition 1
(i) $\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$ for all $a, b \in \mathbb{C}$.
(ii) If $g(x) = f(x + \alpha)$ then $\hat{g}(\omega) = e^{i\alpha\omega}\hat{f}(\omega)$.
(iii) If $h(x) = e^{i\beta x}f(x)$ then $\hat{h}(\omega) = \hat{f}(\omega - \beta)$.

Proposition 2 Suppose that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.
Then (i) $\hat{f}$ is well defined and bounded;
(ii) $\hat{f}$ is continuous;
(iii) $\hat{f}(\omega) \to 0$ as $\omega \to \infty$.

Theorem 1 Let f be a smooth function such that $f, f', f'', \ldots, f^{(k)}$ are all absolutely integrable on $\mathbb{R}$.

Then (i) $\widehat{f^{(k)}}(\omega) = (i\omega)^k \hat{f}(\omega)$;

(ii) $\hat{f}(\omega) = \alpha(\omega)/\omega^k$, where $\lim_{\omega\to\infty} \alpha(\omega) = 0$.

Theorem 2 Let f be a function on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1+|x|^k)|f(x)|\,dx < \infty$ for some integer $k \geq 1$.

Then (i) $\hat{f}$ is k times differentiable;

(ii) $\hat{f}^{(k)}(\omega) = (-i)^k \mathcal{F}[x^k f(x)](\omega)$.

Convolution

Suppose $f, g : \mathbb{R} \to \mathbb{C}$ are bounded, absolutely integrable functions. The function

$$(f * g)(x) = \int_{\mathbb{R}} f(y) g(x - y)\, dy$$

is called the **convolution** of f and g.

Lemma $f * g = g * f$.

Proof: Let $z = x - y$. Then

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y)\, dy$$

$$= \int_{-\infty}^{\infty} f(x - z) g(z)\, dz = (g * f)(x).$$

Convolution Theorem

(i) $\mathcal{F}[f \cdot g] = \mathcal{F}[f] * \mathcal{F}[g]$;

(ii) $\mathcal{F}[f * g] = 2\pi\, \mathcal{F}[f] \cdot \mathcal{F}[g]$.

Proof of (ii): $\quad \mathcal{F}[f * g](\omega) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} (f * g)(x) e^{-i\omega x}\, dx$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(x-y) e^{-i\omega x}\, dx\, dy \qquad (x = y + z)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(z) e^{-i\omega(y+z)}\, dz\, dy = 2\pi\, \hat{f}(\omega) \hat{g}(\omega).$$

Plancherel's Theorem (a.k.a. Parseval's Theorem)

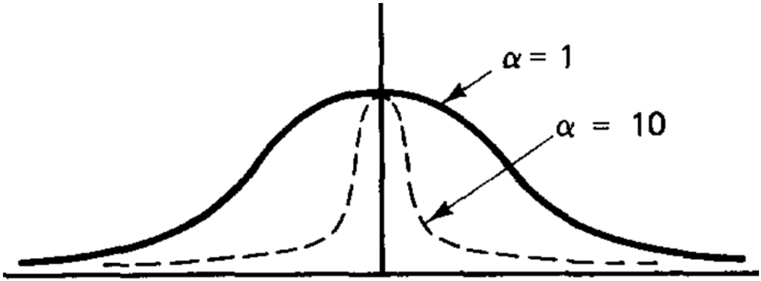
(i) If a function f is both absolutely integrable and square-integrable on $\mathbb{R}$, then $\mathcal{F}[f]$ is also square-integrable. Moreover,

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = 2\pi \int_{\mathbb{R}} |\hat{f}(\omega)|^2 \, d\omega.$$

(ii) If functions f, g are absolutely integrable and square-integrable on $\mathbb{R}$, then

$$\int_{\mathbb{R}} f(x)\overline{g(x)} \, dx = 2\pi \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{g}(\omega)} \, d\omega.$$

That is, $\langle f, g \rangle = 2\pi \, \langle \hat{f}, \hat{g} \rangle$.



Gaussian $g(x) = e^{-\alpha x^2}, \ \ \alpha > 0$

(density of the normal probability distribution)

$$g(x) = e^{-\alpha x^2}, \qquad \hat{g}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-i\omega x}\, dx.$$

$$\hat{g}(\omega) = \frac{1}{\pi}\int_{0}^{\infty} e^{-\alpha x^2} \cos\omega x\, dx.$$

$$\frac{d\hat{g}}{d\omega} = \frac{1}{\pi}\int_{0}^{\infty} e^{-\alpha x^2} \frac{\partial}{\partial\omega}(\cos\omega x)\, dx$$

$$= -\frac{1}{\pi}\int_{0}^{\infty} x e^{-\alpha x^2} \sin\omega x\, dx = \frac{1}{2\alpha\pi}\int_{0}^{\infty} \sin\omega x\, d\left(e^{-\alpha x^2}\right)$$

$$= \frac{e^{-\alpha x^2}\sin\omega x\,\Big|_{x=0}^{\infty}}{2\alpha\pi} - \frac{1}{2\alpha\pi}\int_{0}^{\infty} e^{-\alpha x^2} d(\sin\omega x).$$

$$\frac{d\hat{g}}{d\omega} = -\frac{\omega}{2\alpha\pi}\int_0^\infty e^{-\alpha x^2}\cos\omega x\,dx = -\frac{\omega}{2\alpha}\,\hat{g}(\omega).$$

$$\hat{g}' = -\frac{\omega}{2\alpha}\hat{g} \implies \frac{\hat{g}'}{\hat{g}} = -\frac{\omega}{2\alpha} \implies (\log\hat{g})' = -\frac{\omega}{2\alpha}$$

$$\implies \log\hat{g} = -\frac{\omega^2}{4\alpha} + C \implies \hat{g}(\omega) = ce^{-\omega^2/(4\alpha)},$$

$$\text{where} \quad c = \hat{g}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx.$$

$$\left(2\pi\hat{g}(0)\right)^2 = \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx \int_{-\infty}^{\infty} e^{-\alpha y^2}\,dy$$

$$= \iint_{\mathbb{R}^2} e^{-\alpha x^2} e^{-\alpha y^2}\,dx\,dy = \iint_{\mathbb{R}^2} e^{-\alpha(x^2+y^2)}\,dx\,dy$$

$$= \int_{-\pi}^{\pi}\int_{0}^{\infty} e^{-\alpha r^2} r\,dr\,d\theta = 2\pi \int_{0}^{\infty} e^{-\alpha r^2} r\,dr$$

$$= 2\pi\,\frac{e^{-\alpha r^2}}{-2\alpha}\Big|_{r=0}^{\infty} = \frac{\pi}{\alpha} \qquad \Longrightarrow \quad \hat{g}(0) = \frac{1}{\sqrt{4\pi\alpha}}$$

$$\boxed{g(x) = e^{-\alpha x^2}} \qquad \boxed{\hat{g}(\omega) = \frac{1}{\sqrt{4\pi\alpha}}\,e^{-\omega^2/(4\alpha)}}$$

Heat equation on an infinite interval

Initial value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty),$$

$$u(x,0) = f(x).$$

We assume that f is smooth and rapidly decaying as $x \to \infty$. We search for a solution with the same properties.

Apply the Fourier transform (relative to x) to both sides of the equation:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k\,\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right].$$

Let $U = \mathcal{F}[u]$. That is,

$$U(\omega, t) = \mathcal{F}[u(\cdot, t)](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x}\, dx.$$

Then $\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial U}{\partial t}, \quad \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (i\omega)^2 U(\omega, t).$

Hence $\frac{\partial U}{\partial t} = k(i\omega)^2 U(\omega, t) = -k\omega^2 U(\omega, t).$

General solution: $U(\omega, t) = ce^{-\omega^2 kt}$, where $c = c(\omega)$.

Initial condition $u(x, 0) = f(x)$ implies that $U(\omega, 0) = \hat{f}(\omega)$. Therefore $U(\omega, t) = \hat{f}(\omega)e^{-\omega^2 kt}$.

We know that

$$\mathcal{F}[e^{-\alpha x^2}] = \frac{1}{\sqrt{4\pi\alpha}}\, e^{-\frac{\omega^2}{4\alpha}}, \qquad \alpha > 0.$$

It follows that $e^{-\omega^2 kt} = \mathcal{F}[g(x,t)]$, where

$$g(x,t) = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}, \qquad t > 0.$$

Hence $U(\omega,t) = \hat{f}(\omega)\hat{g}(\omega,t)$. By the convolution theorem, $u = (2\pi)^{-1} f * g$, that is,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tilde{x})g(x-\tilde{x})\, d\tilde{x} \\ &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(\tilde{x})e^{-\frac{(x-\tilde{x})^2}{4kt}}\, d\tilde{x}. \end{aligned}$$

Initial value problem:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty),$$

$$u(x, 0) = f(x).$$

Solution: $u(x, t) = \int_{-\infty}^{\infty} G(x, \tilde{x}, t)\, f(\tilde{x})\, d\tilde{x},$

where $G(x, \tilde{x}, t) = \dfrac{1}{\sqrt{4\pi kt}}\, e^{-\frac{(x-\tilde{x})^2}{4kt}}.$

The solution is in the integral operator form. The function G is called the **kernel** of the operator. Also, $G(x, \tilde{x}, t)$ is called **Green's function** of the problem.