Math 412-501

Theory of Partial Differential Equations

# Lecture 4-1: Green's functions. Dirac delta function.

# Heat equation on the infinite interval

Initial value problem:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty),$$
$$u(x,0) = f(x).$$

**Solution:** $\quad u(x,t) = \int_{-\infty}^{\infty} G(x,\tilde{x},t)\, f(\tilde{x})\, d\tilde{x},$

where $\;G(x,\tilde{x},t) = \dfrac{1}{\sqrt{4\pi kt}}\, e^{-\frac{(x-\tilde{x})^2}{4kt}}.$

The solution is in the integral operator form. The function $G$ is called the **kernel** of the operator. Also, $G(x,\tilde{x},t)$ is called **Green's function** of the problem.

$$\boxed{G(x,\tilde{x},t) = \frac{1}{\sqrt{4\pi kt}}\, e^{-\frac{(x-\tilde{x})^2}{4kt}}}$$

$G(x,\tilde{x},t)$ is:

- defined for $t > 0$,
- positive,
- infinitely smooth.

$$\int_{-\infty}^{\infty} G(x,\tilde{x},t)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}}\, e^{-\frac{(x-\tilde{x})^2}{4kt}}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}}\, e^{-\frac{x^2}{4kt}}\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}\, e^{-y^2}\, dy = 1$$

$$\frac{\partial G(x,\tilde{x},t)}{\partial x} = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\tilde{x})^2}{4kt}} \cdot \frac{-2(x-\tilde{x})}{4kt}$$

$$= -\frac{x-\tilde{x}}{2kt} G(x,\tilde{x},t);$$

$$\frac{\partial^2 G(x,\tilde{x},t)}{\partial x^2} = \left( \frac{(x-\tilde{x})^2}{4k^2t^2} - \frac{1}{2kt} \right) G(x,\tilde{x},t);$$

$$\frac{\partial G(x,\tilde{x},t)}{\partial t} = -\frac{1}{2} t^{-3/2} \cdot \frac{1}{\sqrt{4\pi k}} e^{-\frac{(x-\tilde{x})^2}{4kt}}$$

$$+ \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\tilde{x})^2}{4kt}} \cdot \frac{(x-\tilde{x})^2}{4kt^2} = \left( \frac{(x-\tilde{x})^2}{4kt^2} - \frac{1}{2t} \right) G(x,\tilde{x},t).$$

**Lemma** For any $\tilde{x} \in \mathbb{R}$ the function $u(x,t) = G(x,\tilde{x},t)$ is a solution of the heat equation for $t > 0$.

Besides, $\lim_{t\to 0} G(x,\tilde{x},t) = \begin{cases} \infty & \text{if } x = \tilde{x}, \\ 0 & \text{if } x \neq \tilde{x}. \end{cases}$

We say that $G(x,\tilde{x},t)$ is the solution of the initial value problem

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty),$$

$$u(x,0) = \delta(x - \tilde{x}),$$

where $\delta(x)$ is the **Dirac delta function**.

# Dirac delta function

$\delta(x)$ is a function on $\mathbb{R}$ such that

- $\delta(x) = 0$ for all $x \neq 0$,
- $\delta(0) = \infty$,
- $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

For any continuous function $f$ and any $x_0 \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = f(x_0).$$

$\delta(x)$ is a **generalized function** (or **distribution**).

That is, $\delta$ is a linear functional on a space of test functions $f$ such that $\delta[f] = f(0)$.

## Delta sequence/family

Regular function $g$ can be regarded as a generalized function

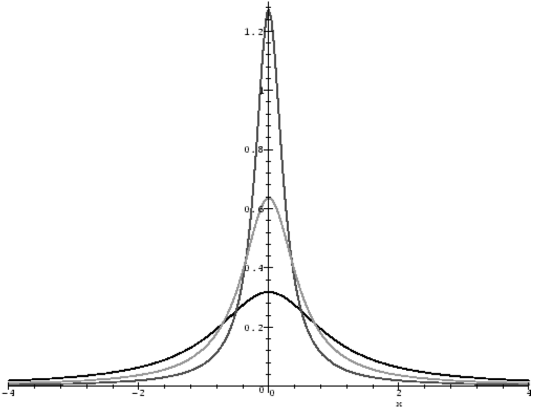
$$f \mapsto \int_{-\infty}^{\infty} f(x)g(x)\,dx.$$

A **delta sequence** is a sequence of (sharply peaked) functions $g_1, g_2, \dots$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)g_n(x)\,dx = f(0)$$

for any test function $f$ (e.g., infinitely smooth and rapidly decaying). That is, $g_n \to \delta$ as $n \to \infty$ (as generalized functions).

A **delta family** is a family of functions $h_\varepsilon$, $0 < \varepsilon \le \varepsilon_0$, such that $\lim_{\varepsilon\to 0} h_\varepsilon = \delta$.




$$h_\varepsilon(x) = \frac{1}{\sqrt{\pi\varepsilon}}\, e^{-x^2/\varepsilon}, \quad \varepsilon > 0.$$

Initial value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty),$$

$$u(x,0) = \delta(x - \tilde{x}).$$

The initial condition means that $g_t(x) = u(x + \tilde{x}, t)$ ought to be a delta family. Indeed, if $u(x,t) = G(x, \tilde{x}, t)$, then

$$g_t(x) = \frac{1}{\sqrt{4\pi kt}}\, e^{-x^2/(4kt)}.$$

Now suppose that

$$u(x,t) = \int_{-\infty}^{\infty} G(x,\tilde{x},t)\, f(\tilde{x})\, d\tilde{x},$$

where $f$ is a test function (smooth and rapidly decaying).

Then $u(x,t)$ is infinitely smooth for $t > 0$ and solves the heat equation. Besides,

$$\lim_{t\to 0} u(x,t) = f(x).$$

Actually, it is sufficient that $f$ be continuous and bounded.

# Heat equation on a semi-infinite interval

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty),$$

$$\frac{\partial u}{\partial x}(0,t) = 0,$$

$$u(x,0) = f(x).$$

**Solution:** $u(x,t) = \displaystyle\int_0^\infty G_1(x,\tilde{x},t)\, f(\tilde{x})\, d\tilde{x},$

where $\quad G_1(x,\tilde{x},t) = \dfrac{1}{\sqrt{4\pi k t}}\left(e^{-\frac{(x-\tilde{x})^2}{4kt}} + e^{-\frac{(x+\tilde{x})^2}{4kt}}\right).$

Clearly, $G_1(x, \tilde{x}, t) = G(x, \tilde{x}, t) + G(-x, \tilde{x}, t)$.
Since $G(x, \tilde{x}, t)$ is a solution of the heat equation for $t > 0$, so are $G(-x, \tilde{x}, t)$ and $G_1(x, \tilde{x}, t)$.
By definition, $G_1(x, \tilde{x}, t)$ is even as a function of $x$.
Therefore

$$\frac{\partial G_1}{\partial x}(0, \tilde{x}, t) = 0.$$

Let $f_1$ denote the even extension of the function $f$ to $\mathbb{R}$, i.e., $f_1(x) = f_1(-x) = f(x)$ for all $x \geq 0$. Then

$$u(x,t) = \int_0^\infty G_1(x, \tilde{x}, t) f(\tilde{x})\, d\tilde{x} = \int_{-\infty}^\infty G(x, \tilde{x}, t) f_1(\tilde{x})\, d\tilde{x}.$$

It follows that $u(x,t)$ is indeed the solution of the initial-boundary value problem. Again, it is sufficient that $f$ be continuous and bounded.

# Heat equation on a finite interval

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L),$$

$$u(0,t) = u(L,t) = 0,$$

$$u(x,0) = f(x) \qquad (0 < x < L).$$

**Solution:** $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n kt} \sin \frac{n\pi x}{L}$,

where $\lambda_n = (n\pi/L)^2$,

$$b_n = \frac{2}{L}\int_0^L f(\tilde{x}) \sin \frac{n\pi \tilde{x}}{L}\, d\tilde{x}.$$

For $t > 0$ we obtain

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-\lambda_n kt} \sin\frac{n\pi x}{L} \int_0^L f(\tilde{x}) \sin\frac{n\pi\tilde{x}}{L}\, d\tilde{x}$$

$$= \int_0^L G_2(x,\tilde{x},t) f(\tilde{x})\, d\tilde{x},$$

where

$$G_2(x,\tilde{x},t) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-\lambda_n kt} \sin\frac{n\pi x}{L} \sin\frac{n\pi\tilde{x}}{L}.$$