Math 412-501

Theory of Partial Differential Equations

Lecture 4-2:
More on the Dirac delta function.
Green's functions for ODEs.

Dirac delta function

$\delta(x)$ is a function on $\mathbb{R}$ such that

- $\delta(x) = 0$ for all $x \neq 0$,
- $\delta(0) = \infty$,
- $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.

For any continuous function f and any $x_0 \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\,dx = f(x_0).$$

$\delta(x)$ is a **generalized function** (or **distribution**).

That is, δ is a linear functional on a space of test functions f such that $\delta[f] = f(0)$.

Distributions

Class of test functions $\mathcal{S}$: consists of infinitely smooth, rapidly decaying functions on $\mathbb{R}$.
To be precise, $f \in \mathcal{S}$ if $\sup |x^k f^{(m)}(x)| < \infty$ for any integers $k, m \geq 0$.

Convergence in $\mathcal{S}$: we say that $f_n \to f$ in $\mathcal{S}$ as $n \to \infty$ if $\sup |x|^k |f_n^{(m)}(x) - f^{(m)}(x)| \to 0$ as $n \to \infty$ for any integers $k, m \geq 0$.

Class of distributions $\mathcal{S}'$: consists of continuous linear functionals on $\mathcal{S}$. That is, a linear map $\ell : \mathcal{S} \to \mathbb{R}$ belongs to $\mathcal{S}'$ if $\ell[f_n] \to \ell[f]$ whenever $f_n \to f$ in $\mathcal{S}$.

Convergence in $\mathcal{S}'$: we say that $\ell_n \to \ell$ in $\mathcal{S}'$ if $\ell_n[f] \to \ell[f]$ for any $f \in \mathcal{S}$.

Examples. (i) Delta function $\delta[f] = f(0)$.

(ii) Shifted δ-function $\delta_{x_0}(x) = \delta(x - x_0)$, $\delta_{x_0}[f] = f(x_0)$.

(iii) Let g be a bounded, locally integrable function on $\mathbb{R}$. Then

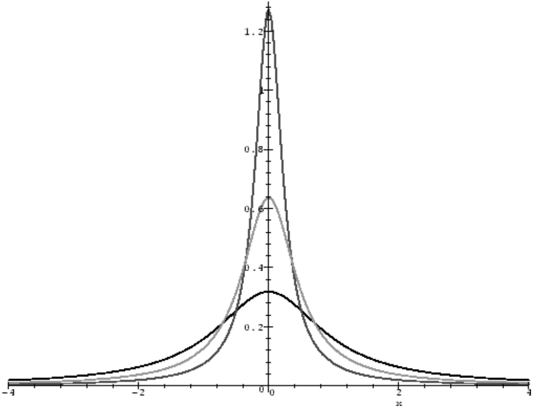
$$f \mapsto \int_{-\infty}^{\infty} f(x)g(x)\,dx$$

is a distribution, which is identified with g.

Delta sequence is a sequence of functions $g_1, g_2, \dots$ such that $g_n \to \delta$ in $\mathcal{S}'$ as $n \to \infty$. That is, for any $f \in \mathcal{S}$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)g_n(x)\,dx = f(0).$$

Delta family is a family of functions h_ε, $0 < \varepsilon \le \varepsilon_0$, such that $\lim_{\varepsilon\to 0} h_\varepsilon = \delta$ in $\mathcal{S}'$.



$$h_\varepsilon(x) = \frac{1}{\sqrt{\pi\varepsilon}}\, e^{-x^2/\varepsilon}, \quad \varepsilon > 0.$$

How to differentiate a distribution

If g is a piecewise differentiable bounded function on $\mathbb{R}$ then

$$\int_{-\infty}^{\infty} f(x)g'(x)\,dx = -\int_{-\infty}^{\infty} f'(x)g(x)\,dx$$

for any test function $f \in \mathcal{S}$.

Let γ be a distribution. Then $\mathcal{S} \ni f \mapsto -\gamma[f']$ is also a distribution, which is denoted γ' and called the **derivative** of γ (in $\mathcal{S}'$).

In the case when γ is a differentiable function, the derivative in $\mathcal{S}'$ coincides with the usual derivative.

Heaviside step function

$$H(x) = \begin{cases} 0 & \text{if } \; x < 0, \\ 1 & \text{if } \; x \geq 0. \end{cases}$$

The Heaviside function is a regular distribution. For any test function $f \in \mathcal{S}$,

$$H'[f] = -\int_{-\infty}^{\infty} f'(x) H(x) \, dx$$

$$= -\int_{0}^{\infty} f'(x) \, dx = -f(x) \Big|_{x=0}^{\infty} = f(0).$$

Thus the derivative of the Heaviside function is the delta function: $H' = \delta$.

Green's functions for ODEs

Boundary value problem:

$$\frac{d^2u}{dx^2} = f(x), \qquad u(0) = u(L) = 0.$$

Definition 1. Green's function of the problem is a function $G(x, x_0)$ $(x, x_0 \in [0, L])$ such that for any f

$$u(x) = \int_0^L f(x_0) G(x, x_0)\, dx_0.$$

Definition 2. Green's function $G(x, x_0)$ of the problem is its solution for $f(x) = \delta(x - x_0)$:

$$\frac{\partial^2 G(x, x_0)}{\partial x^2} = \delta(x - x_0), \qquad G(0, x_0) = G(L, x_0) = 0.$$

Definition 1 shows how to **use** Green's function.
Definition 2 shows how to **find** Green's function.
Both definitions are equivalent.

Definition 2 means that

- $\dfrac{\partial^2 G(x, x_0)}{\partial x^2} = 0$ for $x < x_0$ and $x > x_0$;
- $G(x, x_0)$ is continuous at $x = x_0$;
- $\dfrac{\partial G(x, x_0)}{\partial x}\Big|_{x=x_0+} - \dfrac{\partial G(x, x_0)}{\partial x}\Big|_{x=x_0-} = 1.$

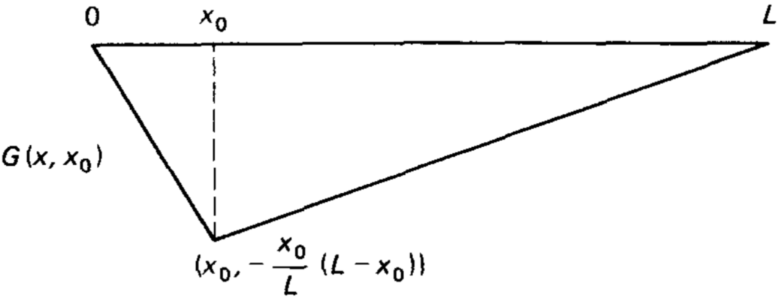
$$G(x, x_0) = \begin{cases} ax + b & \text{if} \quad x < x_0, \\ cx + d & \text{if} \quad x > x_0, \end{cases}$$

where a, b, c, d may depend on x_0.

$$\frac{\partial G(x, x_0)}{\partial x} = \begin{cases} a & \text{if} \quad x < x_0, \\ c & \text{if} \quad x > x_0. \end{cases}$$

Besides, $G(0, x_0) = b$ and $G(L, x_0) = cL + d$.
Therefore

$$\begin{cases} c - a = 1 \\ ax_0 + b = cx_0 + d \\ b = 0 \\ cL + d = 0 \end{cases} \implies \begin{cases} a = (x_0 - L)/L \\ b = 0 \\ c = x_0/L \\ d = -x_0 \end{cases}$$



$$G(x, x_0) = \begin{cases} -\frac{x}{L}(L - x_0) & \text{if} \quad x < x_0, \\ -\frac{x_0}{L}(L - x) & \text{if} \quad x > x_0. \end{cases}$$

$G(x, x_0) = G(x_0, x)$ **(Maxwell's reciprocity)**

Hilbert space $L_2[0, L] = \{h : \int_0^L |h(x)|^2 \, dx < \infty\}$

Dense subspace $\mathcal{H} = \{h \in C^2[0, L] : h(0) = h(L) = 0\}$

Linear operator $\mathcal{L} : \mathcal{H} \to L_2[0, L]$, $\mathcal{L}[h] = h''$

$\mathcal{L}$ is **self-adjoint**: $\langle \mathcal{L}[h], g \rangle = \langle h, \mathcal{L}[g] \rangle$ for all $h, g \in \mathcal{H}$.

$$\langle \mathcal{L}[h], g \rangle = \int_0^L h''(x)\overline{g(x)} \, dx$$

$$= h'(x)\overline{g(x)} \Big|_{x=0}^{L} - \int_0^L h'(x)\overline{g'(x)} \, dx = -\int_0^L h'(x)\overline{g'(x)} \, dx$$

$$= -h(x)\overline{g'(x)} \Big|_{x=0}^{L} + \int_0^L h(x)\overline{g''(x)} \, dx = \langle h, \mathcal{L}[g] \rangle$$

Inverse operator $\mathcal{L}^{-1} : L_2[0, L] \to L_2[0, L]$.

If $\mathcal{L}^{-1}[f] = u$ then $u'' = f$, $u(0) = u(L) = 0$.

$$\mathcal{L}^{-1}[f](x) = \int_0^L G(x, x_0) f(x_0)\, dx_0$$

Since the operator $\mathcal{L}$ is self-adjoint, so is $\mathcal{L}^{-1}$.

$$\langle \mathcal{L}^{-1}[f], g \rangle = \int_0^L \int_0^L G(x, x_0) f(x_0) \overline{g(x)}\, dx_0\, dx$$

$$\langle f, \mathcal{L}^{-1}[g] \rangle = \int_0^L \int_0^L f(x) \overline{G(x, x_0)}\, \overline{g(x_0)}\, dx_0\, dx$$

$\mathcal{L}^{-1}$ is self-adjoint if and only if $G(x, x_0) = \overline{G(x_0, x)}$.

Nonhomogeneous boundary value problem:

$$u''(x) = f(x), \qquad u(0) = \alpha, \ \ u(L) = \beta.$$

We have that $u = u_1 + u_2 + u_3$, where

$$u_1'' = f, \ \ u_1(0) = u_1(L) = 0;$$
$$u_2'' = 0, \ \ u_2(0) = \alpha, \ \ u_2(L) = 0;$$
$$u_3'' = 0, \ \ u_3(0) = 0, \ \ u_3(L) = \beta.$$

It turns out that

$$u_1(x) = \int_0^L G(x, x_0) f(x_0)\, dx_0,$$

$$u_2(x) = \alpha\left(1 - \frac{x}{L}\right) = -\alpha\, \frac{\partial G(x, x_0)}{\partial x_0}\,\Big|_{x_0=0},$$

$$u_3(x) = \beta\, \frac{x}{L} = \beta\, \frac{\partial G(x, x_0)}{\partial x_0}\,\Big|_{x_0=L}.$$

Existense of Green's function

Green's function of an initial/boundary value problem exists only if there is always a **unique** solution.

Example 1. $u'' + u = f, \; u(0) = u(L) = 0.$

Green's function exists if $L \neq n\pi$, $n = 1, 2, \ldots$ (otherwise $u_1(x) = 0$ and $u_2(x) = \sin x$ are both solutions for $f = 0$).

Example 2. $u''(x) + u(x) = f(x), \; 0 < x < L,$ $u(0) = u'(0) = 0.$

Green's function exists for any $L > 0$.