Math 412-501

Theory of Partial Differential Equations

Lecture 4-3: Green's functions for the heat and wave equations.

Distributions

Class of test functions $\mathcal{S}$: consists of infinitely smooth, rapidly decaying functions on $\mathbb{R}$.
To be precise, $f \in \mathcal{S}$ if $\sup |x^k f^{(m)}(x)| < \infty$ for any integers $k, m \geq 0$.

Convergence in $\mathcal{S}$: we say that $f_n \to f$ in $\mathcal{S}$ as $n \to \infty$ if $\sup |x|^k |f_n^{(m)}(x) - f^{(m)}(x)| \to 0$ as $n \to \infty$ for any integers $k, m \geq 0$.

Class of distributions $\mathcal{S}'$: consists of continuous linear functionals on $\mathcal{S}$. That is, a linear map $\ell : \mathcal{S} \to \mathbb{R}$ belongs to $\mathcal{S}'$ if $\ell[f_n] \to \ell[f]$ whenever $f_n \to f$ in $\mathcal{S}$.

Convergence in $\mathcal{S}'$: we say that $\ell_n \to \ell$ in $\mathcal{S}'$ if $\ell_n[f] \to \ell[f]$ for any $f \in \mathcal{S}$.

Examples. (i) Delta function $\delta[f] = f(0)$.

(ii) Shifted δ-function $\delta_{x_0}(x) = \delta(x - x_0)$, $\delta_{x_0}[f] = f(x_0)$.

(iii) Let g be a bounded, locally integrable function on $\mathbb{R}$. Then

$$f \mapsto \int_{-\infty}^{\infty} f(x)g(x)\,dx$$

is a distribution, which is identified with g.

How to differentiate a distribution

Let γ be a distribution. Then $\mathcal{S} \ni f \mapsto -\gamma[f']$ is also a distribution, which is denoted γ' and called the **derivative** of γ (in $\mathcal{S}'$).

If g is a piecewise differentiable bounded function on $\mathbb{R}$ then the derivative in $\mathcal{S}'$ coincides with the usual derivative as

$$\int_{-\infty}^{\infty} f(x)g'(x)\,dx = -\int_{-\infty}^{\infty} f'(x)g(x)\,dx$$

for any test function $f \in \mathcal{S}$.

Example. $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$ (Heaviside step function).
We have that $H' = \delta$.

How to Fourier transform a distribution

$$\mathcal{F}[\delta](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x) e^{-i\omega x}\, dx = \frac{1}{2\pi}.$$

Let g be an absolutely integrable function and f be a test function. Then

$$\int_{-\infty}^{\infty} f(x)\, \mathcal{F}[g](x)\, dx = \int_{-\infty}^{\infty} \mathcal{F}[f](x)\, g(x)\, dx$$

(alternative form of Parseval's identity)

Let γ be a distribution. Then $\mathcal{S} \ni f \mapsto \gamma[\mathcal{F}[f]]$ is also a distribution, which is denoted $\mathcal{F}[\gamma]$ or $\hat{\gamma}$ and called the **Fourier transform** of γ (in $\mathcal{S}'$).

In the case when γ is an absolutely integrable function, both definitions of $\mathcal{F}[\gamma]$ agree.

For any test function $f \in \mathcal{S}$,

$$\hat{\delta}[f] = \delta[\mathcal{F}[f]] = \mathcal{F}[f](0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, dx.$$

Thus $\mathcal{F}[\delta]$ is indeed the constant function $1/(2\pi)$.

Naive approach does not always work:

$$\mathcal{F}^{-1}[1](x) = \int_{-\infty}^{\infty} e^{i\omega x}\, d\omega = ??? \ = 2\pi\, \delta(x).$$

However,

$$g_L(x) = \int_{-L}^{L} e^{i\omega x}\, d\omega = \frac{e^{iLx} - e^{-iLx}}{ix} = \frac{2 \sin Lx}{x},$$

and functions $(2\pi)^{-1} g_L$ form a delta family as $L \to \infty$.

Green's function for the heat equation

Green's function $G(x, t; x_0, t_0)$ for the infinite interval describes heat conduction in an infinite rod caused by an instant point-like heat source which acts at time t_0 at the point x_0 and generates the unit amount of heat energy.

Formally, G solves the equation

$$\frac{\partial G}{\partial t} = k \frac{\partial^2 G}{\partial x^2} + \delta(x - x_0)\,\delta(t - t_0)$$

subject to the condition

$$G(x, t; x_0, t_0) = 0 \quad \text{for} \quad t < t_0.$$

(causality principle)

Apply the Fourier transform (relative to x) to both sides of the equation:

$$\mathcal{F}_x\left[\frac{\partial G}{\partial t}\right] = k\,\mathcal{F}_x\left[\frac{\partial^2 G}{\partial x^2}\right] + \mathcal{F}_x[\delta(x-x_0)]\,\delta(t-t_0).$$

Let $\widehat{G}(\omega, t; x_0, t_0)$ denote the Fourier transform of G relative to x:

$$\widehat{G}(\omega, t; x_0, t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x, t; x_0, t_0) e^{-i\omega x}\,dx.$$

$$\mathcal{F}_x\left[\frac{\partial G}{\partial t}\right] = \frac{\partial \widehat{G}}{\partial t}, \quad \mathcal{F}_x\left[\frac{\partial^2 G}{\partial x^2}\right] = (i\omega)^2\widehat{G} = -\omega^2\widehat{G},$$

$$\mathcal{F}_x[\delta(x-x_0)](\omega) = \frac{1}{2\pi}\,e^{-i\omega x_0}.$$

$$\Longrightarrow \quad \frac{\partial \widehat{G}}{\partial t} = -k\omega^2 \widehat{G} + \frac{e^{-i\omega x_0}}{2\pi}\,\delta(t - t_0).$$

Causality principle implies that

$$\widehat{G}(\omega, t; x_0, t_0) = 0 \quad \text{for} \quad t < t_0.$$

It follows that

$$\widehat{G}(\omega, t; x_0, t_0) = \begin{cases} 0 & \text{for} \quad t < t_0, \\ c(\omega, x_0, t_0) e^{-k\omega^2 t} & \text{for} \quad t > t_0; \end{cases}$$

$$\widehat{G}(\omega, t; x_0, t_0)\Big|_{t=t_0+} - \widehat{G}(\omega, t; x_0, t_0)\Big|_{t=t_0-} = \frac{e^{-i\omega x_0}}{2\pi}.$$

Hence $c(\omega, x_0, t_0) = \dfrac{e^{-i\omega x_0}}{2\pi}\, e^{k\omega^2 t_0}.$

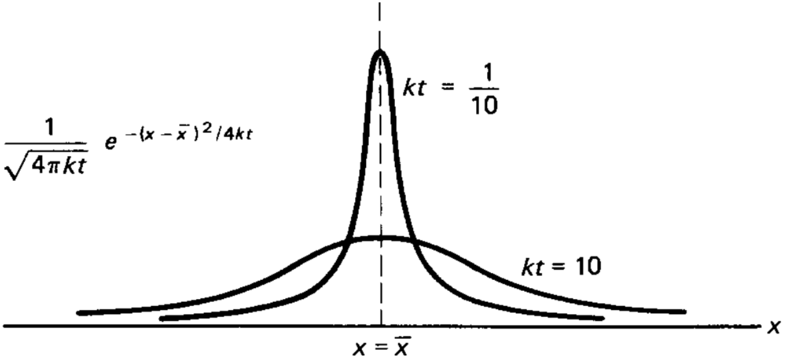
Then $\widehat{G}(\omega, t; x_0, t_0) = \dfrac{e^{-i\omega x_0}}{2\pi}\, e^{-k\omega^2(t-t_0)}$ for $t > t_0$.

So for $t > t_0$ we obtain

$$G(x, t; x_0, t_0) = \int_{-\infty}^{\infty} \frac{e^{-i\omega x_0}}{2\pi}\, e^{-k\omega^2(t-t_0)}\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k\omega^2(t-t_0)}\, e^{i\omega(x-x_0)}\, d\omega$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{k(t-t_0)}}\, e^{-\frac{(x-x_0)^2}{4k(t-t_0)}} = \frac{1}{\sqrt{4\pi k(t-t_0)}}\, e^{-\frac{(x-x_0)^2}{4k(t-t_0)}}.$$



$\frac{1}{\sqrt{4\pi kt}} e^{-(x-\bar{x})^2/4kt}$
$kt = \frac{1}{10}$
$kt = 10$
x
$x = \bar{x}$

General nonhomogeneous problem

Initial value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (-\infty < x < \infty,\ t > 0),$$

$$u(x,0) = f(x).$$

Solution: $u(x,t) =$

$$= \int_0^\infty \int_{-\infty}^\infty G(x,t;x_0,t_0)\,Q(x_0,t_0)\,dx_0\,dt_0$$

$$+ \int_{-\infty}^\infty G(x,t;x_0,0)\,f(x_0)\,dx_0.$$

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Initial value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (-\infty < x < \infty,\ t > 0),$$

$$u(x,0) = f(x).$$

Solution: $u(x,t) =$

$$= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-t_0)}}\, e^{-\frac{(x-x_0)^2}{4k(t-t_0)}}\, Q(x_0,t_0)\, dx_0\, dt_0$$

$$+ \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}}\, e^{-\frac{(x-x_0)^2}{4kt}}\, f(x_0)\, dx_0.$$

Green's function for the wave equation

Green's function $G(x, t; x_0, t_0)$ for the infinite interval describes vibrations of an infinite string caused by an instant unit force which is applied at time t_0 to the point x_0.

Formally, G solves the equation

$$\frac{\partial^2 G}{\partial t^2} = c^2 \frac{\partial^2 G}{\partial x^2} + \delta(x - x_0)\,\delta(t - t_0)$$

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(causality principle)