Math 412-501

Theory of Partial Differential Equations

Lecture 4-5:
Uniqueness of solutions of PDEs.
The maximum principle.

Uniqueness of solutions of PDEs

Principal idea: under some natural, non-restrictive conditions the initial/boundary value problems for the heat, wave, and Laplace's equations have unique solutions.

Theorem The initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad (0 < x < L,\ 0 < t < T),$$

$$u(x,0) = f(x) \qquad (0 < x < L),$$

$$u(0,t) = A(t), \;\; u(L,t) = B(t) \qquad (0 < t < T)$$

has at most one solution that is twice differentiable on $[0, L] \times [0, T]$.

Proof: Suppose u_1 and u_2 are two solutions.
Let $w = u_1 - u_2$. Then

$$\frac{\partial w}{\partial t} = k\,\frac{\partial^2 w}{\partial x^2} \qquad (0 < x < L,\ 0 < t < T),$$

$$w(x,0) = 0, \qquad\qquad w(0,t) = w(L,t) = 0.$$

Let $E(t) = \int_0^L |w(x,t)|^2\,dx$. Then

$$E'(t) = 2\int_0^L w\,\frac{\partial w}{\partial t}\,dx = 2k\int_0^L w\,\frac{\partial^2 w}{\partial x^2}\,dx$$

$$= 2kw\,\frac{\partial w}{\partial x}\bigg|_{x=0}^{L} - 2k\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \le 0.$$

Since $E(t) \ge 0$ and $E(0) = 0$, it follows that

$$E = 0 \implies w = 0 \implies u_1 = u_2.$$

The maximum principle

Theorem Let $D \subset \mathbb{R}^2$ be a bounded domain and $u : \overline{D} \to \mathbb{R}$ be a continuous function on the closure $\overline{D} = D \cup \partial D$.

If u is twice differentiable in D and $\nabla^2 u \geq 0$ then

$$\max_{\mathbf{x} \in \overline{D}} u(\mathbf{x}) = \max_{\mathbf{x} \in \partial D} u(\mathbf{x}).$$

(the maximum is attained on the boundary)

Corollary 1 If $\nabla^2 u = 0$ then the maximum and the minimum of u in $\overline{D}$ are both attained on the boundary ∂D.

Proof: Since $\nabla^2 u = 0$, the theorem applies to both u and $-u$.

Corollary 2 If $\nabla^2 u = 0$ and $u = 0$ on the boundary ∂D, then $u = 0$ in D as well.

Corollary 3 Given functions $Q : D \to \mathbb{R}$ and $f : \partial D \to \mathbb{R}$, the boundary value problem

$$\nabla^2 u = Q \quad \text{in the domain} \quad D,$$
$$u = f \quad \text{on the boundary} \quad \partial D$$

has at most one solution that is twice differentiable in D and continuous on the closure $\overline{D}$.

Proof: Suppose u_1 and u_2 are two solutions. Let $w = u_1 - u_2$. Then $\nabla^2 w = 0$ in D and $w = 0$ on ∂D. By Corollary 2, $w = 0$ in D, i.e., $u_1 = u_2$.

Proof of the maximum principle

Lemma 1 Let $f : (a, b) \longrightarrow \mathbb{R}$ be a twice differentiable function. If f has a local maximum at a point $c \in (a, b)$, then $f'(c) = 0$, $f''(c) \leq 0$.

Lemma 2 Let u be a twice differentiable function on the domain D. If $\nabla^2 u > 0$ then u has no local maximum in D.

Proof: Suppose that u has a local maximum at some point $(x_0, y_0) \in D$. Then the function $f(x) = u(x, y_0)$ has a local maximum at x_0 while $g(y) = u(x_0, y)$ has a local maximum at y_0.
By Lemma 1, $f''(x_0) \leq 0$, $g''(y_0) \leq 0$. Then $\nabla^2 u(x_0, y_0) = f''(x_0) + g''(y_0) \leq 0$, a contradiction.

Proof of Theorem: Suppose u is continuous on $\overline{D}$ and $\nabla^2 u \geq 0$ in D. Let $w(x,y) = x^2 + y^2$. Then $\nabla^2 w = 4$. For any $\varepsilon > 0$ let $u_\varepsilon = u + \varepsilon w$. Then $\nabla^2 u_\varepsilon = \nabla^2 u + 4\varepsilon > 0$ in D.

By Lemma 2, u_ε has no local maximum in D. Hence

$$\sup_D u_\varepsilon \leq \max_{\partial D} u_\varepsilon.$$

But $\max_{\partial D} u_\varepsilon \leq \max_{\partial D} u + \varepsilon \max_{\partial D} w$ and $\sup_D u \leq \sup_D u_\varepsilon$. Therefore

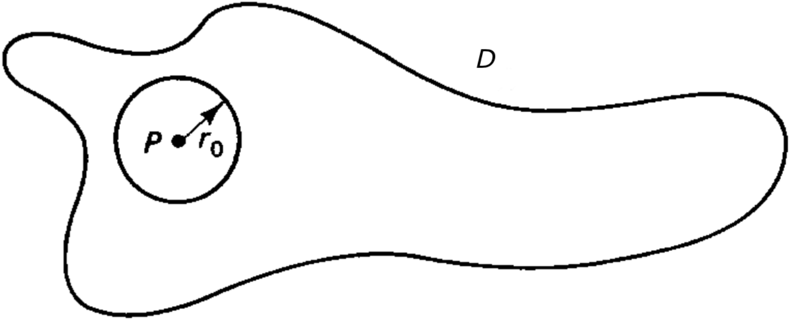
$$\sup_D u \leq \max_{\partial D} u + \varepsilon \max_{\partial D} w.$$

Since ε can be chosen arbitrarily small, we have

$$\sup_D u \leq \max_{\partial D} u.$$

Mean value theorem

The value of a harmonic function at any point P is the average of its values along any circle centered at P.



$$\nabla^2 u = 0 \text{ in } D \implies u(P) = \frac{1}{2\pi r_0} \oint_{C(P,r_0)=\{\mathbf{x}:|\mathbf{x}-P|=r_0\}} u(\mathbf{x})\, ds$$

Proof: Introduce the polar coordinates r, θ with the origin at P. Let $f(\theta) = u(r_0, \theta)$, $-\pi < \theta \leq \pi$. Then u is the solution of the boundary value problem

$$\nabla^2 u = 0 \qquad (0 \leq r < r_0),$$
$$u(r_0, \theta) = f(\theta).$$

Solution:

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^n (a_n \cos n\theta + b_n \sin n\theta),$$

where $a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$ is the Fourier series of $f(\theta)$.

Now $u(P) = u(0,\theta) = a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(\theta)\, d\theta.$