MATH 415 Spring 2021

## Sample problems for Exam 2

Any problem may be altered, removed or replaced by a different one!

**Problem 1.** Let M be the set of all  $2\times 2$  matrices of the form  $\binom{n-k}{0-n}$ , where n and k are rational numbers. Under the operations of matrix addition and multiplication, does this set form a ring? Does M form a field?

**Problem 2.** Let L be the set of the following  $2\times 2$  matrices with entries from the field  $\mathbb{Z}_2$ :

$$A = \begin{pmatrix} [0] & [0] \\ [0] & [0] \end{pmatrix}, \quad B = \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix}, \quad C = \begin{pmatrix} [1] & [1] \\ [1] & [0] \end{pmatrix}, \quad D = \begin{pmatrix} [0] & [1] \\ [1] & [1] \end{pmatrix}.$$

Under the operations of matrix addition and multiplication, does this set form a ring? Does L form a field?

**Problem 3.** Prove that for a ring with unity, commutativity of addition follows from the other axioms. [Hint: simplify the expression (1+1)(x+y) in two different ways.]

**Problem 4.** Find a direct product of cyclic groups that is isomorphic to  $G_{16}$  (multiplicative group of all invertible elements of the ring  $\mathbb{Z}_{16}$ ).

**Problem 5.** Determine the last two digits of 303<sup>303</sup>.

**Problem 6.** Find all integer solutions of the equation 21x - 32y = 4.

**Problem 7.** Find all integer solutions of the equation 2x + 3y + 5z = 7.

**Problem 8.** Solve the equation  $2x^{100} + x^{71} + x^{29} = 0$  over the field  $\mathbb{Z}_{11}$ .

**Problem 9.** Factor a polynomial  $p(x) = x^3 - 3x^2 + 3x - 2$  into irreducible factors over the field  $\mathbb{Z}_7$ .

**Problem 10.** Factor a polynomial  $p(x) = x^4 + x^3 - 2x^2 + 3x - 1$  into irreducible factors over the field  $\mathbb{Q}$ . [Hint: since p has integer coefficients, there exists a factorization such that each factor has integer coefficients.]