MATH 415

Modern Algebra I

Lecture 2:
Binary operations.

Binary operations

Definition. A **binary operation** $*$ on a nonempty set S is simply a function $* : S \times S \to S$.

The usual notation for the element $*(x, y)$ is $x * y$.

The pair $(S, *)$ is called a **binary algebraic structure**.

Examples: arithmetic operations

Addition $+$ of:

natural numbers, integers, rationals, real numbers, complex numbers, vectors, matrices of fixed dimensions, real-valued functions with fixed domain.

Subtraction $-$ of:

all above examples with addition except for natural numbers.

Multiplication $\times$ of:

natural numbers, integers, rationals, real numbers, complex numbers, roots of unity, quaternions, square matrices of fixed dimensions, real-valued functions with fixed domain.

Division $/$ of:

positive rationals, nonzero rationals, positive real numbers, nonzero real numbers, nonzero complex numbers, roots of unity, invertible diagonal matrices of fixed dimensions.

Examples: addition modulo n

Given a natural number n, let
$\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$.
A binary operation $+_n$ (**addition modulo** n) on $\mathbb{Z}_n$ is defined for any $x, y \in \mathbb{Z}_n$ by

$$x +_n y = \begin{cases} x + y & \text{if } x + y < n, \\ x + y - n & \text{if } x + y \geq n. \end{cases}$$

Now let n be a positive real number and $\mathbb{R}_n = [0, n)$. The binary operation $+_n$ on $\mathbb{R}_n$ is defined by the same formula as above.

Remark. The binary structure $(\mathbb{R}_{2\pi}, +_{2\pi})$ is an abstract model for rotations of a circle.

Examples: composition of functions

Let $F(X, X)$ denote the set of all functions $f : X \to X$. Given two functions $f, g \in F(X, X)$, the composition $f \circ g$ is another function in $F(X, X)$ defined by $(f \circ g)(x) = f(g(x))$, $x \in X$.

Then $\circ$ is a binary operation on the following subsets of $F(X, X)$:

- all functions,
- all invertible functions,
- all injective functions,
- all surjective functions.

Examples: set theory

$\mathcal{P}(X)$ = the set of all subsets of some set X.

Binary operations on $\mathcal{P}(X)$:

- union $A \cup B$,
- intersection $A \cap B$,
- set difference $A \setminus B$,
- symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Examples: logic

Binary logic $\mathcal{L} = \{\text{“true”}, \text{“false”}\}$.

Binary operations on $\mathcal{L}$:

- logical AND,
- logical OR,
- XOR (eXclusive OR),
- $\Longrightarrow$,
- $\Longleftarrow$,
- $\Longleftrightarrow$.

Overall, there are $2^{2 \cdot 2} = 16$ distinct binary operations on any set consisting of two elements.

Counterexamples

- Reciprocal of a positive number.

$S = \mathbb{R}_+, \; *(x) = x^{-1}.$

This operation is unary, not binary.

- Mean arithmetic value of three numbers.

$$S = \mathbb{R}, \; *(x, y, z) = \frac{x + y + z}{3}.$$

This operation is ternary, not binary.

- Division of real numbers.

$S = \mathbb{R}, \; x * y = x/y.$

The operation is only partially defined as one cannot divide by 0.

Counterexamples

- Division of natural numbers.

$S = \mathbb{N},\ x * y = x/y.$

The operation is not well defined as $x * y$ is not always an integer.

- Solution of a quadratic equation.

$S = \mathbb{C},\ (x * y)^2 + x(x * y) + y = 0.$

The operation is not defined uniquely as the equation can have two solutions. In other words, this is not a function.

Restriction

Suppose $(S, *)$ is a binary structure. If S_0 is a nonempty set then we can restrict $*$, as a function, to $S_0 \times S_0$.

If the restricted function is a binary operation on S_0 then we call it the **restriction** of the operation to S_0 and use the same notation $*$.

The restricted function is a binary operation on S_0 if and only if the subset S_0 is **closed under the operation** $*$ which means that $x, y \in S_0$ implies $x * y \in S_0$. Otherwise the restricted operation is not well defined.

Useful properties of binary operations

Suppose $(S, *)$ is a binary structure.

- Commutativity:

$g * h = h * g$ for all $g, h \in S$.

- Associativity:

$(g * h) * k = g * (h * k)$ for all $g, h, k \in S$.

- Existence of the identity element:

there exists an element $e \in S$ such that $e * g = g * e = g$ for all $g \in S$.

- Existence of the inverse element:

for any $g \in S$ there exists an element $h \in S$ such that $g * h = h * g = e$ (where e is the identity element).

- Cancellation:

$g * h_1 = g * h_2$ implies $h_1 = h_2$ and $h_1 * g = h_2 * g$ implies $h_1 = h_2$ for all $g, h_1, h_2 \in S$.

Cayley table

A binary operation on a finite set can be given by a **Cayley table** (i.e., "multiplication" table):

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

The Cayley table is convenient to check commutativity of the operation (the table should be symmetric relative to the diagonal), cancellation properties (left cancellation holds if each row contains all elements, right cancellation holds if each column contains all elements), existence of the identity element, and existence of the inverse.

However this table is not convenient to check associativity of the operation.

Problem. The following is a partially completed Cayley table for a certain commutative operation with cancellation:

$*$	a	b	c	d
a	b			c
b			c	
c				a
d		d		

Complete the table.

Solution:

$*$	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	b	a
d	c	d	a	b