MATH 415–501 Fall 2021

Homework assignment #12

Problem 1. Let $\phi : R \to R'$ be an isomorphism of rings and suppose I is a two-sided ideal in R. Prove that the factor ring R/I is isomorphic to the factor ring $R'/\phi(I)$.

Problem 2 (2 pts). Consider the ring of functions $\mathcal{F}(\mathbb{Z}, \mathbb{R})$. Let I be the set of those functions $h : \mathbb{Z} \to \mathbb{R}$ that are zero everywhere except at finitely many points, i.e., $\mathbb{Z} \setminus h^{-1}(0)$ is a finite set.

(i) Show that I is an ideal in $\mathcal{F}(\mathbb{Z}, \mathbb{R})$.

(ii) Show that the ideal I is not maximal.

Problem 3 (2 pts). Let $\mathbb{F}$ be a field. Given two polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and $q(x) = b_0 + b_1x + \cdots + b_mx^m$ in $\mathbb{F}[x]$, the *formal composition* $p \circ q$ is a polynomial given by $(p \circ q)(x) = a_0 + a_1q(x) + a_2(q(x))^2 + \cdots + a_n(q(x))^n$.

(i) Let $q \in \mathbb{F}[x]$. Show that the map $\phi_q : \mathbb{F}[x] \to \mathbb{F}[x]$ given by $\phi_q(p) = p \circ q$ is a homomorphism of rings.

(ii) In the case $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$, show that any homomorphism of the ring $\mathbb{F}[x]$ to itself is either identically zero or of the form ϕ_q for some $q \in \mathbb{F}[x]$.

Problem 4 (2 pts). Let $\mathbb{F}$ be a field. For any polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ in $\mathbb{F}[x]$ we define a function $f_p : \mathbb{F} \to \mathbb{F}$ by $f_p(c) = a_0 + a_1c + \cdots + a_nc^n$ for all $c \in \mathbb{F}$.

(i) Show that $f_{p\circ q} = f_p \circ f_q$ for all $p, q \in \mathbb{F}[x]$, where $p \circ q$ is the formal composition of polynomials and $f_p \circ f_q$ is the usual composition of functions.

(ii) Prove that formal composition is an associative operation on $\mathbb{F}[x]$ in the case $\mathbb{F}$ is infinite. [Hint: use the fact that $p \mapsto f_p$ is a one-to-one map in the case $\mathbb{F}$ is infinite.]

(iii) **(+2 pts)** Prove that formal composition is an associative operation on $\mathbb{F}[x]$ for any field $\mathbb{F}$.

Problem 5 (3 pts). Let $\mathbb{F}$ be a field.

(i) Show that polynomials of degree 1 in $\mathbb{F}[x]$ form a group under the operation of formal composition.

(ii) In the case $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$, show that any automorphism ϕ of the ring $\mathbb{F}[x]$ is given by $\phi(p(x)) = p(\alpha x + \beta)$, where $\alpha, \beta \in \mathbb{F}$, $\alpha \neq 0$.

(iii) In the case $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$, prove that the group $\mathrm{Aut}(\mathbb{F}[x])$ of automorphisms of the ring $\mathbb{F}[x]$ is isomorphic to the following subgroup of $GL(2, \mathbb{F})$:

$$\left\{ \begin{pmatrix} 1 & \beta \\ 0 & \alpha \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{F},\ \alpha \neq 0 \right\}.$$