MATH 415–501 Fall 2021

Homework assignment #2

Problem 1. Define a binary operation $*$ on $\mathbb{R}$ such that the function $\phi(x) = 1 - x$, $x \in \mathbb{R}$ is an isomorphism of the binary structure $(\mathbb{R}, \cdot)$ with $(\mathbb{R}, *)$. Use it to show that "$x \cdot 0 = 0$ for all $x \in \mathbb{R}$" is not a structural property of $(\mathbb{R}, \cdot)$.

Problem 2. Consider the following binary operations on the set $S = GL(n, \mathbb{R})$ of invertible $n \times n$ matrices with real entries: right division $A \,/_r\, B = AB^{-1}$ and left division $A \,/_\ell\, B = B^{-1}A$. Prove that the binary structures $(S, /_r)$ and $(S, /_\ell)$ are isomorphic.

Problem 3. There are 16 different binary operations on the set $S = \{a, b\}$. How many non-isomorphic binary structures are there? Give one example from each isomorphy class (use the Cayley tables).

Problem 4. Given an integer $n > 0$, let $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$ be the set of all integers divisible by n. Prove that $(n\mathbb{Z}, +)$ is a group and that it is isomorphic to $(\mathbb{Z}, +)$.

Problem 5. Let $S = \mathbb{R} \setminus \{-1\}$ and define a binary operation $*$ on S by $x * y = x + y + xy$. Prove that $(S, *)$ is a group.

Problem 6. Let $(G, *)$ be a group. Suppose that $g * g = e$ for all $g \in G$, where e is the identity element. Prove that the group is abelian.

Problem 7. Let $(G, *)$ be a finite group. Suppose that $g * g \neq e$ for all $g \neq e$, where e is the identity element. Prove that the number of elements in G is odd.

Problem 8. Suppose $(S, *)$ is a group and define a binary operation $\bullet$ on S by $x \bullet y = y * x$. Prove that $(S, \bullet)$ is a group and that it is isomorphic to $(S, *)$.

Problem 9. A square matrix A with real entries is called *orthogonal* if $A^T = A^{-1}$. Prove that all $n \times n$ orthogonal matrices form a group under matrix multiplication.

Problem 10. A square matrix is called *unipotent* if it is upper-triangular and all diagonal entries are equal to 1. Prove that all $n \times n$ unipotent matrices with real entries form a group under matrix multiplication.