MATH 415

Modern Algebra I

Lecture 17:
Transformation groups.

Transformation groups

Definition. A **transformation group** is a group where elements are bijective transformations of a fixed set X and the operation is composition.

Examples.

- Symmetric group S_X: all bijective functions $f : X \to X$.
- Translations of the real line: $T_c(x) = x + c$, $x \in \mathbb{R}$.
- $\mathrm{Homeo}(\mathbb{R})$: the group of all invertible functions $f : \mathbb{R} \to \mathbb{R}$ such that both f and f^{-1} are continuous (such functions are called **homeomorphisms**).
- $\mathrm{Homeo}^+(\mathbb{R})$: the group of all increasing functions in $\mathrm{Homeo}(\mathbb{R})$ (those that preserve orientation of the real line).
- $\mathrm{Diff}(\mathbb{R})$: the group of all invertible functions $f : \mathbb{R} \to \mathbb{R}$ such that both f and f^{-1} are continuously differentiable (such functions are called **diffeomorphisms**).

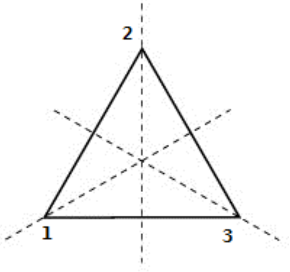
Groups of symmetries

Definition. A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a **motion** (or a **rigid motion**) if it preserves distances between points.

Theorem All motions of $\mathbb{R}^n$ form a transformation group. Any motion $f : \mathbb{R}^n \to \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix ($A^T A = AA^T = I$).

Given a geometric figure $F \subset \mathbb{R}^n$, a **symmetry** of F is a motion of $\mathbb{R}^n$ that preserves F. All symmetries of F form a transformation group.

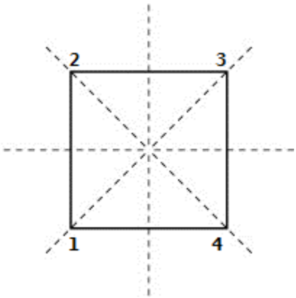
Example. • The **dihedral group** D_n is the group of symmetries of a regular n-gon. It consists of $2n$ elements: n reflections, $n-1$ rotations by angles $2\pi k/n$, $k = 1, 2, \dots, n-1$, and the identity function.



Equlateral triangle

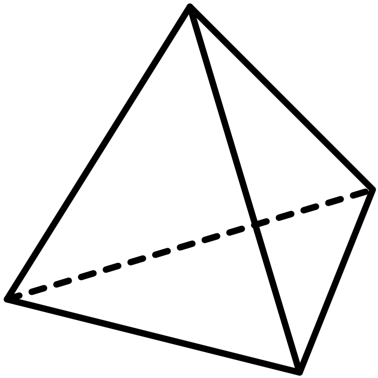
Any symmetry of a polygon maps vertices to vertices. Therefore it induces a permutation on the set of vertices. Moreover, the symmetry is uniquely recovered from the permutation.

In the case of the equilateral triangle, any permutation of vertices comes from a symmetry.



Square

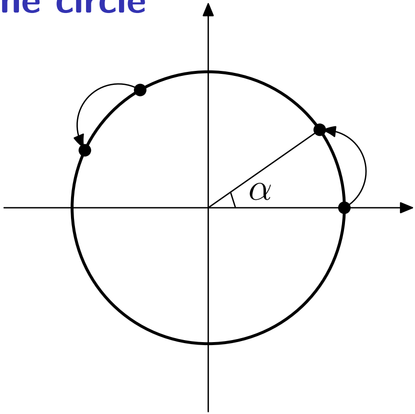
In the case of the square, not every permutation of vertices comes from a symmetry of the square. The reason is that a symmetry must map adjacent vertices to adjacent vertices.



Regular tetrahedron

Any symmetry of a polyhedron maps vertices to vertices. In the case of the regular tetrahedron, any permutation of vertices comes from a symmetry.

Rotations of the circle



Let $R_\alpha : S^1 \to S^1$ be the rotation of the circle S^1 by angle $\alpha \in \mathbb{R}$. All rotations R_α, $\alpha \in \mathbb{R}$ form a transformation group. Namely, $R_\alpha R_\beta = R_{\alpha+\beta}$, $R_\alpha^{-1} = R_{-\alpha}$, and $R_0 = \mathrm{id}$.

The group of rotations is a subgroup of the group of all symmetries of the circle (the other symmetries are reflections).

Group of automorphisms

Definition. Any isomorphism of a group G onto itself is called an **automorphism** of G.

Automorphisms are "symmetries" of the group as an algebraic structure. All automorphisms of a given group G form a transformation group denoted $\mathrm{Aut}(G)$.

Example. • Conjugation.

Take any $g \in G$ and define a map $i_g : G \to G$ by $i_g(x) = gxg^{-1}$ for all $x \in G$. Then $i_g(xy) = g(xy)g^{-1}$ $= gx(g^{-1}g)yg^{-1} = (gxg^{-1})(gyg^{-1}) = i_g(x)i_g(y)$. Hence i_g is a homomorphism. Further, $i_g(i_h(x)) = i_g(hxh^{-1})$ $= g(hxh^{-1})g^{-1} = (gh)x(gh)^{-1} = i_{gh}(x)$. Hence $i_g \circ i_h = i_{gh}$ for all $g, h \in G$. In particular, $i_g \circ i_{g^{-1}} = i_{g^{-1}} \circ i_g = i_e = \mathrm{id}_G$. Therefore $i_{g^{-1}} = (i_g)^{-1}$ so that i_g is bijective.

Automorphisms of the form i_g are called **inner**. They form a group $\mathrm{Inn}(G)$, which is a normal subgroup of $\mathrm{Aut}(G)$.