Sample problems for Exam 2

Any problem may be altered, removed or replaced by a different one!

Problem 1. Let M be the set of all 2×2 matrices of the form $\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}$, where n and k are rational numbers. Under the operations of matrix addition and multiplication, does this set form a ring? Does M form a field?

Problem 2. Let L be the set of the following 2×2 matrices with entries from the field $\mathbb{Z}_2$:

$$A = \begin{pmatrix} [0] & [0] \\ [0] & [0] \end{pmatrix}, \quad B = \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix}, \quad C = \begin{pmatrix} [1] & [1] \\ [1] & [0] \end{pmatrix}, \quad D = \begin{pmatrix} [0] & [1] \\ [1] & [1] \end{pmatrix}.$$

Under the operations of matrix addition and multiplication, does this set form a ring? Does L form a field?

Problem 3. Prove that for a ring with unity, commutativity of addition follows from the other axioms. [Hint: simplify the expression $(1+1)(x+y)$ in two different ways.]

Problem 4. Find a direct product of cyclic groups that is isomorphic to G_{16} (multiplicative group of all invertible elements of the ring $\mathbb{Z}_{16}$).

Problem 5. Determine the last two digits of 303^{303}.

Problem 6. Find all integer solutions of the equation $21x - 32y = 4$.

Problem 7. Find all integer solutions of the equation $2x + 3y + 5z = 7$.

Problem 8. Solve the equation $2x^{100} + x^{71} + x^{29} = 0$ over the field $\mathbb{Z}_{11}$.

Problem 9. Factor a polynomial $p(x) = x^3 - 3x^2 + 3x - 2$ into irreducible factors over the field $\mathbb{Z}_7$.

Problem 10. Factor a polynomial $p(x) = x^4 + x^3 - 2x^2 + 3x - 1$ into irreducible factors over the field $\mathbb{Q}$. [Hint: since p has integer coefficients, there exists a factorization such that each factor has integer coefficients.]