## Sample problems for the final exam

## Any problem may be altered or replaced by a different one!

Problem 1. For any positive integer $n$ let $n \mathbb{Z}$ denote the set of all integers divisible by $n$.
(i) Does the set $3 \mathbb{Z} \cup 4 \mathbb{Z} \cup 7 \mathbb{Z}$ form a semigroup under addition? Does it form a group?
(ii) Does the set $3 \mathbb{Z} \cup 4 \mathbb{Z} \cup 7 \mathbb{Z}$ form a semigroup under multiplication? Does it form a group?

Problem 2. Consider a relation $\sim$ on a group $G$ defined as follows. For any $g, h \in G$ we let $g \sim h$ if and only if $g$ is conjugate to $h$, which means that $g=x h x^{-1}$ for some $x \in G$ (where $x$ may depend on $g$ and $h$ ). Show that $\sim$ is an equivalence relation on $G$.

Problem 3. Find all subgroups of the group $G_{15}$ (multiplicative group of invertible congruence classes modulo 15.)

Problem 4. Let $\pi=(12)(23)(34)(45)(56), \sigma=(123)(234)(345)(456)$. Find the order and the sign of the following permutations: $\pi, \sigma, \pi \sigma$, and $\sigma \pi$.

Problem 5. Let $G$ be a group. Suppose $H$ is a subgroup of $G$ of finite index $(G: H)$. Further suppose that $K$ is a subgroup of $H$ of finite index $(H: K)$. Prove that $K$ is a subgroup of finite index in $G$ and, moreover, $(G: K)=(G: H)(H: K)$.

Problem 6. Let $G$ be the group of all symmetries of a regular tetrahedron $T$. The group $G$ naturally acts on the set of vertices of $T$, the set of edges of $T$, and the set of faces of $T$.
(i) Show that each of the three actions is transitive.
(ii) Show that the stabilizer of any vertex is isomorphic to the symmetric group $S_{3}$.
(iii) Show that the stabilizer of any edge is isomorphic to the Klein 4 -group $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$.
(iv) Show that the stabilizer of any face is isomorphic to $S_{3}$.

Problem 7. Let $S$ be a nonempty set and $\mathcal{P}(S)$ be the set of all subsets of $S$.
(i) Prove that $\mathcal{P}(S)$ with the operations of symmetric difference $\triangle$ (as addition) and intersection $\cap$ (as multiplication) is a commutative ring with unity.
(ii) Prove that the ring $\mathcal{P}(S)$ is isomorphic to the ring of functions $\mathcal{F}\left(S, \mathbb{Z}_{2}\right)$.

Problem 8. Solve a system of congruences (find all solutions):

$$
\left\{\begin{array}{l}
x \equiv 2 \bmod 5 \\
x \equiv 3 \bmod 6 \\
x \equiv 6 \bmod 7
\end{array}\right.
$$

Problem 9. Find all integer solutions of a system

$$
\left\{\begin{array}{l}
2 x+5 y-z=1 \\
x-2 y+3 z=2
\end{array}\right.
$$

[Hint: eliminate one of the variables.]

Problem 10. Factor a polynomial $p(x)=x^{4}-2 x^{3}-x^{2}-2 x+1$ into irreducible factors over the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_{5}$ and $\mathbb{Z}_{7}$.
[Hint: notice that $p(x)=x^{4} p(1 / x)$.]

Problem 11. Let

$$
M=\left\{\left.\left(\begin{array}{ll}
x & 0 \\
y & z
\end{array}\right) \right\rvert\, x, y, z \in \mathbb{R}\right\}, \quad J=\left\{\left.\left(\begin{array}{ll}
0 & 0 \\
y & 0
\end{array}\right) \right\rvert\, y \in \mathbb{R}\right\} .
$$

(i) Show that $M$ is a subring of the matrix ring $\mathcal{M}_{2,2}(\mathbb{R})$.
(ii) Show that $J$ is a two-sided ideal in $M$.
(iii) Show that the factor ring $M / J$ is isomorphic to $\mathbb{R} \times \mathbb{R}$.

Problem 12. The polynomial $f(x)=x^{6}+3 x^{5}-5 x^{3}+3 x-1$ has how many distinct complex roots?
[Hint: multiple roots of $f$ are also roots of the derivative $f^{\prime}$.]

