MATH 415–502 Fall 2022

Homework assignment #6

Problem 1. Abelian groups of order 72 form how many isomorphism classes? Give an example for each class.

Problem 2 (2 pts). Suppose $G = \{e, a, b, c\}$ is a non-cyclic group of order 4 (where e is the identity element).

(i) Prove that $a^2 = b^2 = c^2 = abc = e$.

(ii) Show that any bijective map $f : G \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $f(e) = (0, 0)$ is an isomorphism of groups. [Hint: compare the Cayley tables.]

Problem 3 (3 pts). Suppose G is a non-abelian group of order 6.

(i) Prove that G has an element of order 2 and an element of order 3.
[Hint: Lagrange's theorem alone does not imply this; use Problems 6 and 7 from HW #2.]

(ii) Let a be any element of order 2, b be any element of order 3, and e be the identity element. Show that $G = \{e, b, b^2, a, ab, ab^2\}$.

(iii) Prove that $ba = ab^2$ and $b^2a = ab$.

(iv) Show that there is an isomorphism $f : G \to S_3$ such that $f(a) = (1\,2)$ and $f(b) = (1\,2\,3)$.
[Hint: compare the Cayley graphs.]

Problem 4. Let $\mathrm{Inn}(G)$ denote the set of all inner automorphisms of a group G. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of the group $\mathrm{Aut}(G)$ of all automorphisms of G.

Problem 5. Prove that $\mathrm{Inn}(G) \cong G/Z(G)$, where $Z(G)$ is the center of the group G.

Problem 6. Prove that $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$.
[Hint: consider the action of $\mathrm{Aut}(G)$ on elements of order 2; also, use Problem 2(ii).]

Problem 7. Prove that $\mathrm{Aut}(S_3) \cong S_3$.
[Hint: consider the action of $\mathrm{Aut}(G)$ on elements of order 2; also, use Problem 3 or Problem 5.]