MATH 415–502 Fall 2022

Homework assignment #7

Problem 1. Let R be a ring. Prove that $x^2 - y^2 = (x - y)(x + y)$ for all $x, y \in R$ if and only if the ring R is commutative.

Problem 2. Show that the set $\{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\}$, equipped with the usual addition and multiplication, is a ring.

Problem 3. Show that the set $\{p + q\sqrt{2} \mid p, q \in \mathbb{Q}\}$, equipped with the usual addition and multiplication, is a field.

Problem 4 (2 pts). An element x of a ring R is called *nilpotent* if $x^n = 0$ for some integer $n \geq 1$ (where n can depend on x). Prove that the set of all nilpotent elements of a commutative ring R is a sub-ring.

Problem 5. Prove that a ring R has no nonzero nilpotent elements if and only if $x = 0$ is the only solution of the equation $x^2 = 0$ in R.

Problem 6. Find an example of a ring that has divisors of zero but does not have nonzero nilpotent elements.

Problem 7. An element x of a ring R is called *idempotent* if $x^2 = x$. Prove that any domain with unity has at most two idempotent elements.

Problem 8 (2 pts). A ring R is called *Boolean* if every element of R is idempotent. Prove that every Boolean ring is commutative.